

A'CAMPO CURVATURE BUMPS AND THE DIRAC PHENOMENON NEAR A SINGULAR POINT

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ABSTRACT. The level curves of an analytic function germ almost always have bumps at unexpected points near the singularity. This profound discovery of N. A'Campo is fully explored in this paper for $f(z, w) \in \mathbb{C}\{z, w\}$, using the Newton-Puiseux infinitesimals and the notion of gradient canyon. Equally unexpected is the Dirac phenomenon: as $c \rightarrow 0$, the total Gaussian curvature of $f(z, w) = c$ accumulates in the gradient canyons.

1. INTRODUCTION

Let $f(x, y) \in \mathbb{R}\{x, y\}$ be a real analytic function germ, $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. The level curves $f = c$, $0 < |c| < \epsilon$, have “bumps” near 0, as we all know.

The following two examples, shown in Fig.1, are illustrative:

$$f_2(x, y) = \frac{1}{2}x^2 - \frac{1}{3}y^3, \quad f_4(x, y) = \frac{1}{4}x^4 - \frac{1}{5}y^5.$$

We all know $f_2 = c$ attains maximum curvature when crossing the y -axis. However, a profound discovery of N. A'Campo is that this is rather an isolated case.

For example, the curvature of $f_4 = c$ is actually 0 on the y -axis; the maximum is attained instead as the level curve crosses $x = \pm ay^{4/3} + \dots$, $a = (2/7)^{1/6}$.

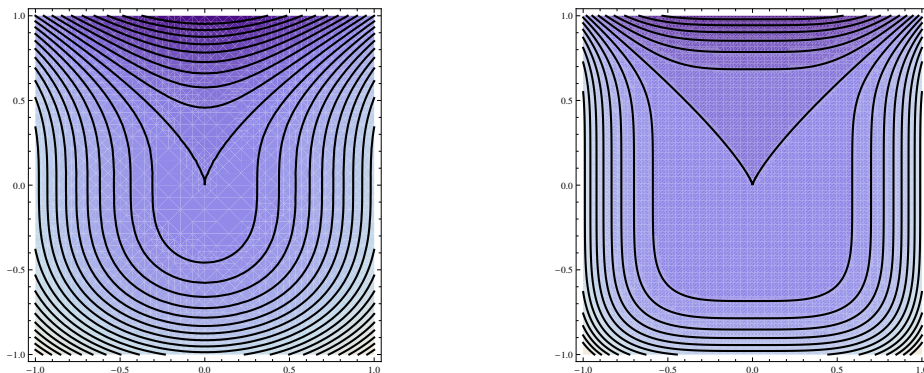


FIGURE 1. $f_2 = c$, $f_4 = c$

Date: March 12, 2013.

2000 Mathematics Subject Classification. Primary 14HXX, 32SXX, 53AXX Secondary 58XX.

Key words and phrases. Singular points of plane curves, Curvature, Newton-Puiseux Infinitesimals, Dirac phenomenon.

We explore this idea in the complex case, using the language of *Newton-Puiseux infinitesimals* (defined in [8], [9], recalled below) and the notion of “*gradient canyon*”.

The real case, with some subtle differences, is studied in [5] and a forthcoming paper.

For earlier explorations in the complex case see Langevin [10] (total curvature), Barosso-Teissier [1] (concentration of curvature), Siersma-Tibar [13] (Gauss-Bonnet).

Take $f \in \mathbb{C}\{z, w\}$. A level curve $\mathcal{S}_c : f = c$ is a Riemann Surface in \mathbb{C}^2 , having Gaussian curvature (see §8 (I))

$$K(z, w) = \frac{2|\Delta_f(z, w)|^2}{(|f_z|^2 + |f_w|^2)^3}, \quad \Delta_f(z, w) := \begin{vmatrix} f_{zz} & f_{zw} & f_z \\ f_{wz} & f_{ww} & f_w \\ f_z & f_w & 0 \end{vmatrix}. \quad (1.1)$$

(This is actually the *negative* of the usual Gaussian curvature defined in text books.)

Take a holomorphic map germ

$$\alpha : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, 0), \quad \alpha(t) \neq 0.$$

The image set germ $\alpha_* := \text{Im}(\alpha)$ is an irreducible curve germ in \mathbb{C}^2 , having a unique tangent $T(\alpha_*)$ at 0, $T(\alpha_*) \in \mathbb{C}P^1$. We call α_* a *Newton-Puiseux infinitesimal* at $T(\alpha_*)$.

The *Enriched Riemann Sphere* is $\mathbb{C}P_*^1 := \{\alpha_*\}$, furnished with the structures defined in §2. The image of $t \mapsto (at, bt)$ is identified with $[a : b] \in \mathbb{C}P^1$; hence $\mathbb{C}P^1 \subset \mathbb{C}P_*^1$.

The curvature *computed along* α_* , if not zero, can be written as

$$K(\alpha(t)) = as^L + \cdots, \quad a > 0, \quad L \in \mathbb{Q} \text{ (rational numbers)},$$

where $s = s(t)$ is the arc length, $s(0) = 0$. This is dominated by the term as^L as $s \rightarrow 0$.

If $K \equiv 0$ along α_* , we write $(a, L) := (0, \infty)$. Hence we introduce the notations

$$(a, L) := a\delta^L, \quad 0_\nu := 0\delta^\infty, \quad \mathcal{V}(\mathbb{R}) := \{a\delta^L \mid a \neq 0\} \cup \{0_\nu\}.$$

where δ is a symbol.

A *lexicographic ordering* on $\mathcal{V}(\mathbb{R})$ is defined: 0_ν is the smallest element, and

$$a\delta^L > a'\delta^{L'} \text{ if and only if either } L < L', \text{ or else } L = L', a > a'.$$

We also define $a\delta^L \gg a'\delta^{L'}$ (*substantially larger than*) if $L < L'$. Thus, for example,

$$2\delta^{3/2} > \delta^{3/2} \gg 10^{10}\delta^2 \gg 0_\nu.$$

The *curvature function* K_* on $\mathbb{C}P_*^1$, and the component L_* are defined as follows:

$$K_* : \mathbb{C}P_*^1 \rightarrow \mathcal{V}(\mathbb{R}), \quad \alpha_* \mapsto a\delta^L; \quad L_* : \mathbb{C}P_*^1 \rightarrow \mathbb{Q}, \quad \alpha_* \mapsto L. \quad (1.2)$$

We shall define the *A'Campo bumps* of K_* , and show how to compute them in Theorem A. Theorem B asserts that every bump is a local maximum of K_* . In Theorem C the total curvature over a gradient canyon, as $c \rightarrow 0$, is given by a Gauss-Bonnet type formula; the integral behaves like that of a Dirac function, this is Theorem D. The polars are perturbed within the canyons to create twin networks of iterated torus knots; in Theorem E the Milnor number μ_f is expressed as their linking number (to measure the entanglement).

We first study K_* on $\mathbb{C}P_*^1$ (the space of analytic arcs at 0), then return to \mathbb{C}^2 . The Newton polygons and the Newton-Puiseux coordinates play a vital role in the proofs.

Theorems A, B, C have been announced in [4].

To exclude the trivial cases, we shall assume $O(f) \geq 2$, and $K_* \neq \text{const}$ (see §8 (II)).

2. STRUCTURES ON $\mathbb{C}P_*^1$ AND MAIN RESULTS

Recall that the classical Newton-Puiseux Theorem asserts that the field \mathbb{F} of convergent fractional power series in an indeterminate y is algebraically closed. ([15], [16].)

A non-zero element of \mathbb{F} is a (finite or infinite) convergent series

$$\alpha(y) = a_0 y^{n_0/N} + \cdots + a_i y^{n_i/N} + \cdots, \quad n_0 < n_1 < \cdots, \quad n_i \in \mathbb{Z}, \quad (2.1)$$

where $0 \neq a_i \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $\text{GCD}(N, n_0, n_1, \dots) = 1$. The *conjugates* of α are

$$\alpha_{\text{conj}}^{(k)}(y) := \sum a_i \theta^{kn_i} y^{n_i/N}, \quad 0 \leq k \leq N-1, \quad \theta := e^{\frac{2\pi\sqrt{-1}}{N}}.$$

The *order* is $O_y(\alpha) := n_0/N$, $O_y(0) := +\infty$. The *Puiseux multiplicity* is $m_{\text{puis}}(\alpha) := N$.

The following \mathbb{D}_0 is an integral domain having quotient field \mathbb{F} , ideals $\mathbb{D}_1, \mathbb{D}_{1+}$:

$$\mathbb{D}_0 := \{\alpha \in \mathbb{F} | O_y(\alpha) \geq 0\}, \quad \mathbb{D}_1 := \{\alpha | O_y(\alpha) \geq 1\}, \quad \mathbb{D}_{1+} := \{\alpha | O_y(\alpha) > 1\}.$$

The *pointwise convergence topology* on \mathbb{D}_1 is defined as follows. Rewrite (2.1) as

$$\alpha(y) = \sum_{q \in \mathbb{Q}} c_\alpha(q) y^q, \quad c_\alpha(q) := \begin{cases} a_i & \text{if } q = n_i/N, i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Take $\varepsilon := \{\varepsilon_1, \dots, \varepsilon_s\}$, $\varepsilon_i > 0$, $\sigma := \{q_1, \dots, q_s\}$, $q_i \geq 1$. A *neighbourhood* of α is, by definition,

$$\mathcal{N}_{\varepsilon, \sigma}^{\text{pwc}}(\alpha) := \{\beta \in \mathbb{D}_1 | |c_\beta(q_i) - c_\alpha(q_i)| < \varepsilon_i, 1 \leq i \leq s\}. \quad (2.2)$$

As in Projective Geometry, $\mathbb{C}P_*^1$ is the union of two charts: $\mathbb{C}P_*^1 = \mathbb{C}_* \cup \mathbb{C}'_*$,

$$\mathbb{C}_* := \{\beta_* \in \mathbb{C}P_*^1 | T(\beta_*) \neq [1 : 0]\}, \quad \mathbb{C}'_* := \{\beta_* \in \mathbb{C}P_*^1 | T(\beta_*) \neq [0 : 1]\}.$$

Take $\alpha \in \mathbb{D}_1$ in (2.1). The map germ, abusing notation,

$$\alpha : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, 0), \quad t \mapsto (\alpha(t^N), t^N),$$

is holomorphic. Hence $\alpha_* \in \mathbb{C}_*$; all conjugates of $\alpha(y)$ give the same α_* .

The *Newton-Puiseux coordinate system* on \mathbb{C}_* is, by definition, the surjection

$$\pi : \mathbb{D}_1 \longrightarrow \mathbb{C}_*, \quad \alpha \mapsto \alpha_*.$$

When α_* is given, the conjugate class of α is unique. Moreover,

$$T(\alpha_*) = [0 : 1] \text{ if and only if } \alpha \in \mathbb{D}_{1+}.$$

The pointwise convergence topology on \mathbb{D}_1 induces one on \mathbb{C}_* :

$$\mathcal{N}_{\varepsilon, \sigma}^{\text{pwc}}(\alpha_*) := \{\beta_* | \beta \in \mathcal{N}_{\varepsilon, \sigma}^{\text{pwc}}(\alpha)\}.$$

The topology on \mathbb{C}'_* is similarly defined; that on $\mathbb{C}P_*^1$ is generated by these topologies.

To define the *contact order* $\mathcal{C}_{ord}(\alpha_*, \beta_*)$, we can assume $\alpha_*, \beta_* \in \mathbb{C}_*$. Then

$$\mathcal{C}_{ord}(\alpha_*, \beta_*) := \begin{cases} \infty & \text{if } \alpha_* = \beta_*, \\ \max_{i,j} \{O_y(\alpha_{conj}^{(i)}(y) - \beta_{conj}^{(j)}(y))\} & \text{if } \alpha_* \neq \beta_*. \end{cases}$$

The *horn subspaces* of $\mathbb{C}P_*^1$ centred at α_* of *orders* e, e^+ are

$$\mathcal{H}_e(\alpha_*) := \{\beta_* \mid \mathcal{C}_{ord}(\alpha_*, \beta_*) \geq e\}, \quad \mathcal{H}_{e^+}(\alpha_*) := \{\beta_* \mid \mathcal{C}_{ord}(\alpha_*, \beta_*) > e\},$$

respectively. In particular,

$$\mathcal{C}_{ord}(\alpha_*, \beta_*) = 1 \text{ if } T(\alpha_*) \neq T(\beta_*); \quad \mathcal{H}_1(\alpha_*) = \mathbb{C}P_*^1 \text{ for all } \alpha_*.$$

When there is no need to specify α_* , we write $\mathcal{H}_e := \mathcal{H}_e(\alpha_*)$.

Let $\mathcal{H}_e := \mathcal{H}_e(\alpha_*)$ be given. Take $\eta(y) := \alpha(y) + uy^e$, $u \in \mathbb{C}$ *generic*. Then

$$L_*^{grc}(\mathcal{H}_e) := L_*(\eta_*) \tag{2.3}$$

is a *well-defined constant*. See Remark 4.1.

Definition 2.1. A horn subspace $\mathcal{H}_e(\alpha_*)$ is a *curvature tableland* if

- (1) $\beta_* \in \mathcal{H}_e(\alpha_*) \implies L_*(\beta_*) \geq L_*^{grc}(\mathcal{H}_e(\alpha_*))$; and
- (2) in the case $e > 1$, there exists e' , $1 \leq e' < e$, such that

$$\nu_* \in \mathcal{H}_{e'}(\alpha_*) - \mathcal{H}_e(\alpha_*) \implies L_*(\nu_*) > L_*^{grc}(\mathcal{H}_e(\alpha_*)).$$

Example. For $z^2 - w^3$, $\mathcal{H}_2(0_*)$ is a curvature tableland,

$$L_*^{grc}(\mathcal{H}_2(0_*)) = -4 = L_*(\beta_*) \text{ for all } \beta_* \in \mathcal{H}_2(0_*).$$

For $z^4 - w^5$, $\mathcal{H}_{4/3}(0_*)$ is a curvature tableland, where

$$L_*^{grc}(\mathcal{H}_{4/3}(0_*)) = -\frac{8}{3} < L_*(0_*) = \infty, \quad K_*(0_*) = 0_{\mathcal{V}}.$$

We shall see in Theorem A how to find all curvature tablelands.

A *horn interval* of *radius* r , $r > 0$, is, by definition,

$$\mathcal{H}_e(\alpha_*, r) := \{\beta_* \mid \beta(y) = \alpha(y) + (cy^e + \cdots), \ |c| \leq r\}.$$

Definition 2.2. Let \mathcal{H}_e be a curvature tableland. Take $\beta_* \in \mathcal{H}_e$. We say K_* has an *A'Campo bump* on $\mathcal{H}_{e^+}(\beta_*)$, or simply say $\mathcal{H}_{e^+}(\beta_*)$ is an *A'Campo bump*, if there exists $\epsilon > 0$,

$$\mu_* \in \mathcal{H}_e(\beta_*, \epsilon) \implies K_*(\beta_*) \geq K_*(\mu_*). \tag{2.4}$$

In the following, consider a given $f(z, w)$. We can apply a generic unitary transformation, if necessary, so that f is *mini-regular* in z , that is,

$$f(z, w) := H_m(z, w) + H_{m+1}(z, w) + \cdots, \quad H_m(1, 0) \neq 0,$$

where $m = O(f)$, $H_k(z, w)$ a homogeneous k -form. Let us also write

$$H_m(z, w) = c(z - z_1 w)^{m_1} \cdots (z - z_r w)^{m_r}, \quad m_i \geq 1, \quad z_i \neq z_j \text{ if } i \neq j, \tag{2.5}$$

where $1 \leq r \leq m$, $\sum m_i = m$, $c \neq 0$. Thus $H_m(z, w)$ is *degenerate* if and only if $r < m$.

Let ζ_i denote the Newton-Puiseux roots of $f(z, w)$, and γ_j those of f_z :

$$f(z, w) = \text{unit} \cdot \prod_{i=1}^m (z - \zeta_i(w)), \quad f_z(z, w) = \text{unit} \cdot \prod_{j=1}^{m-1} (z - \gamma_j(w)), \quad (2.6)$$

where $O_w(\zeta_i) \geq 1$, $O_w(\gamma_j) \geq 1$; γ_j, γ_{j*} are called *polars*.

Definition 2.3. Given a polar γ . Let $d_{gr}(\gamma)$ denote the *smallest* number e such that

$$O_w(\| \text{Grad } f(\gamma(w), w) \|) = O_w(\| \text{Grad } f(\gamma(w) + uw^e, w) \|), \quad (2.7)$$

where $u \in \mathbb{C}$ is a *generic* number. We call $d_{gr}(\gamma)$ the *gradient degree* of γ .

The *gradient canyon* of γ in \mathbb{D}_1 , and the *gradient canyon* of γ_* in \mathbb{C}_* are

$$\mathcal{G}(\gamma) := \{ \alpha \in \mathbb{D}_1 \mid O_y(\alpha - \gamma) \geq d_{gr}(\gamma) \}, \quad \mathcal{G}_*(\gamma_*) := \mathcal{H}_{d_{gr}(\gamma)}(\gamma_*),$$

respectively. When there is no confusion, we simply call them “canyons”, and write

$$d := d_{gr}(\gamma), \quad \mathcal{G} := \mathcal{G}(\gamma), \quad \mathcal{G}_* := \mathcal{G}_*(\gamma_*).$$

We also call $d(\mathcal{G}_*) := d$ the *degree* of \mathcal{G} and \mathcal{G}_* ; the *multiplicity* of $\mathcal{G}, \mathcal{G}_*$ are

$$m(\mathcal{G}) := \# \{ k \mid \mathcal{G}(\gamma_k) = \mathcal{G} \}, \quad m(\mathcal{G}_*) := \# \{ k \mid \mathcal{G}_*(\gamma_{k*}) = \mathcal{G}_* \},$$

respectively. Finally, we say $\mathcal{G}(\gamma)$ and $\mathcal{G}_*(\gamma_*)$ are *minimal* if

$$\mathcal{G}(\gamma_j) \subseteq \mathcal{G}(\gamma) \implies \mathcal{G}(\gamma_j) = \mathcal{G}(\gamma).$$

(The gradient canyons are not topological invariants; they are invariants of a stronger notion of equi-singularity, to be studied in another paper.)

Example 2.4. Take $f = z^m - w^n$, $2 \leq m \leq n$. There is only one polar $\gamma = 0$,

$$d = \frac{n-1}{m-1}, \quad \mathcal{G} = \{ uy^{\frac{n-1}{m-1}} + \dots \mid u \in \mathbb{C} \}, \quad m(\mathcal{G}) = m - 1.$$

The following example shows that in general $m(\mathcal{G}) \neq \# \{ k \mid \gamma_k \in \mathcal{G}(\gamma) \}$, and there exist non-minimal canyons. However, these only happen in the case $d = 1$; see Addendum 2.6.

Example. Take $g = z^4 - 2z^2w^2 - w^{100}$, $\gamma_1 = 0$, $\gamma_2, \gamma_3 = \pm w$. Then $d_{gr}(\gamma_1) = 97$, $d_{gr}(\gamma_2) = 1$,

$$\mathcal{G}(\gamma_1) \subset \mathcal{G}(\gamma_2) = \mathcal{G}(\gamma_3) = \mathbb{D}_1, \quad m(\mathcal{G}(\gamma_2)) = 2 \neq \# \{ k \mid \gamma_k \in \mathcal{G}(\gamma_2) \} = 3.$$

Here $\mathcal{G}(\gamma_1)$ is minimal, but $\mathcal{G}(\gamma_2), \mathcal{G}(\gamma_3)$ are not.

Take a polar γ with $d < \infty$. We define $L_\gamma \in \mathbb{Q}$ and a rational function $R_\gamma(u)$, $u \in \mathbb{C}$.

We can assume $\gamma \in \mathbb{D}_{1+}$ so that $T(\gamma_*) = [0 : 1]$. If $d > 1$, define $L_\gamma, R_\gamma(u)$ by

$$K(\gamma + uy^d, y) = 2R_\gamma(u)y^{2L_\gamma} + \dots, \quad R_\gamma(u) \not\equiv 0, \quad (2.8)$$

where y can be considered as the arc length of $(\gamma + uy^d)_*$ since $\lim y/s = 1$. Note that

$$L_\gamma = \frac{1}{2} L_*((\gamma + uy^d)_*) \text{ for generic } u.$$

If $d = 1$, define L_γ and $R_\gamma(u)$ by

$$K(\gamma + \frac{uy}{\sqrt{1+|u|^2}}, \frac{y}{\sqrt{1+|u|^2}}) = 2R_\gamma(u)y^{2L_\gamma} + \dots, \quad R_\gamma(u) \not\equiv 0. \quad (2.9)$$

Lemma 2.5. *The function $R_\gamma(u)$, $1 \leq d < \infty$, is defined and continuous for all $u \in \mathbb{C}$,*

$$R_\gamma(u) \geq 0, \quad \lim_{u \rightarrow \infty} R_\gamma(u) = 0, \quad L_\gamma = -d. \quad (2.10)$$

Hence the absolute maximum of $R_\gamma(u)$ is attained. (There may be many local maxima.)

Theorem A. *A minimal canyon in \mathbb{C}_* with $d < \infty$ is a curvature tableland, and vice versa.*

Take a minimal canyon $\mathcal{G}(\gamma)$ (in \mathbb{D}_1) with $d < \infty$. Take a local maximum $R_\gamma(c)$ of $R_\gamma(u)$. Let $\gamma^{+c}(y) := \gamma(y) + cy^d$. Then $\mathcal{H}_{d+}(\gamma_^{+c})$ is an A'Campo bump.*

All A'Campo bumps can be found in this way.

When $d = \infty$, γ is a multiple root of f , $\mathcal{G}(\gamma) = \{\gamma\}$, no A'Campo bump arises from γ .

Example. For $f_2(z, w) = \frac{1}{2}z^2 - \frac{1}{3}w^3$, there is only one polar $\gamma = 0$, having $d = 2$,

$$R_\gamma(u) = (|u|^2 + 1)^{-3}, \quad L_\gamma = -2, \quad K(\gamma + uy^d, y) = 2R_\gamma(u)y^{-4} + \cdots.$$

In this example, $R_\gamma(u)$ is maximum at $u = 0$, as expected.

Next, consider $f_4(z, w) = \frac{1}{4}z^4 - \frac{1}{5}w^5$, having $\gamma = 0$, $d = \frac{4}{3}$,

$$\Delta = z^2w^3(4z^4 - 3w^5), \quad R_\gamma(u) = \frac{9|u|^4}{(|u|^6 + 1)^3}, \quad L_\gamma = -d = -\frac{4}{3}.$$

Here $R_\gamma(0) = 0$, $R_\gamma(u)$ is maximum on the circle $|u| = (2/7)^{1/6}$, like a volcanic ring.

Addendum 2.6. *Every gradient canyon $\mathcal{G}(\gamma)$ with $d > 1$ is minimal. In this case,*

$$m(\mathcal{G}) = \#\{k \mid \gamma_k \in \mathcal{G}\}, \quad m(\mathcal{G}_*) = \#\{k \mid \gamma_{k*} \in \mathcal{G}_*\}. \quad (2.11)$$

Let r be as in (2.5). There are $r - 1$ polars of gradient degree 1; moreover,

$$d_{gr}(\gamma) = 1 \implies \mathcal{G}(\gamma) = \mathbb{D}_1, \quad m(\mathcal{G}(\gamma)) = r - 1. \quad (2.12)$$

A minimal $\mathcal{G}(\gamma)$ with $d = 1$ exists if and only if $H_m(z, w)$ is non-degenerate. In this case every polar has $d = 1$, \mathbb{CP}_^1 is the only curvature tableland and the only gradient canyon.*

We now define the *perturbation topology* on \mathbb{CP}_*^1 . Take ε, σ as in (2.2). Let

$$C_{\varepsilon, \sigma} := \{a_1 \delta^{q_1} \mid \varepsilon_1 \leq a_1 < \infty\} \cup \cdots \cup \{a_s \delta^{q_s} \mid \varepsilon_s \leq a_s < \infty\}, \quad C_{\varepsilon, \sigma} \subset \mathcal{V}(\mathbb{R}).$$

Then a *neighbourhood* of μ_* in \mathbb{C}_* is, by definition,

$$\mathcal{N}_{\varepsilon, \sigma}^{ptb}(\mu_*) := \{\mu_*\} \cup \{\nu_* \mid \nu(y) - \mu(y) = ay^q + \cdots, \quad a \neq 0, \quad |a|\delta^q \notin C_{\varepsilon, \sigma}\}.$$

The topology on \mathbb{C}'_* is similarly defined; that on \mathbb{CP}_*^1 is generated by these two.

Theorem B. *Every μ_* in an A'Campo bump $\mathcal{H}_{d+}(\gamma_*^{+c})$ is a local maximum of K_* in the perturbation topology, hence also in the pointwise convergence topology.*

More specifically, K_ is constant on $\mathcal{H}_{d+}(\gamma_*^{+c})$, and there exists $\mathcal{N}_{\varepsilon, \sigma}^{ptb}(\mu_*)$,*

- (1) $\alpha_* \in [\mathcal{H}_d(\mu_*) - \mathcal{H}_{d+}(\mu_*)] \cap \mathcal{N}_{\varepsilon, \sigma}^{ptb}(\mu_*) \implies K_*(\mu_*) > K_*(\alpha_*), \quad L_*(\mu_*) = L_*(\alpha_*);$
- (2) $\alpha_* \in \mathcal{N}_{\varepsilon, \sigma}^{ptb}(\mu_*) - \mathcal{H}_d(\mu_*) \implies K_*(\mu_*) \gg K_*(\alpha_*).$

It is clear that $\mathcal{N}_{\varepsilon, \sigma}^{pwc}(\mu_*) \subset \mathcal{N}_{\varepsilon, \sigma}^{ptb}(\mu_*)$. A local maximum in the perturbation topology is therefore also one in the pointwise convergence topology.

We now carefully return to \mathbb{C}^2 . Consider $\mathcal{S}_c := \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = c\}$ and the disk

$$D_\eta := \{(z, w) \in \mathbb{C}^2 \mid \sqrt{|z|^2 + |w|^2} \leq \eta\}, \quad \eta > 0 \text{ sufficiently small.}$$

A horn interval $\mathcal{H}_e(\alpha_*, r)$ gives rise to a horn-shaped compact subset of D_η :

$$I(\alpha_*, e, r; \eta) := \{(z, w) \in \beta_* \cap D_\eta \mid \beta(y) = \alpha(y) + uy^e, |u| \leq r\},$$

and then the germ at $\eta = 0$:

$$I(\alpha_*, e, r) := \text{germ of } I(\alpha_*, e, r; \eta) \text{ as } \eta \rightarrow 0.$$

We call $I(\alpha_*, e, r)$ an *infinitesimal interval* (or disk) of order e , radius r .

Definition 2.7. The *total (asymptotic Gaussian) curvature* of \mathcal{S}_c over $I(\alpha_*, e, r)$ is

$$\int_{I(\alpha_*, e, r)} KdS := \lim_{\eta \rightarrow 0} \left\{ \lim_{c \rightarrow 0} \int_{\mathcal{S}_c \cap I(\alpha_*, e, r; \eta)} KdS \right\}, \quad S := \text{surface area.}$$

(The order of taking limit is important: Fix η , let $c \rightarrow 0$, then let $\eta \rightarrow 0$.)

Take a polar γ with $1 < d < \infty$. The *total curvature* over $\mathcal{G}_* := \mathcal{G}_*(\gamma_*)$ is, by definition,

$$\lim_{R \rightarrow \infty} \int_{I(\gamma_*, d, R)} KdS, \quad \text{to be written also as } \int_{\mathcal{G}_*} KdS. \quad (2.13)$$

Finally, the *Milnor number* of f on \mathcal{G}_* is, by definition,

$$\mu_f(\mathcal{G}_*) := \sum_j [O_y(f(\gamma_j(y)), y) - 1], \quad \text{summing over all } j, \gamma_{j*} \in \mathcal{G}_*. \quad (2.14)$$

Theorem C. (Compare [10], [1].) *Let γ be a polar, $1 < d_{gr}(\gamma) < \infty$. Then*

$$\int_{\mathcal{G}_*} KdS = 2\pi[\mu_f(\mathcal{G}_*) + m(\mathcal{G}_*)], \quad \mathcal{G}_* := \mathcal{G}_*(\gamma_*). \quad (2.15)$$

The integral vanishes on every substantially smaller infinitesimal interval. That is to say, for any $\epsilon > 0$ (however small), and any R (however large),

$$\int_{I(\gamma_*, d+\epsilon, R)} KdS = 0, \quad d := d_{gr}(\gamma). \quad (2.16)$$

Formula (2.16) remains true when $d = 1$.

Take a polar γ_* , with tangent $\tau_* \in \mathbb{CP}^1$. We shall see in Lemma 3.1 that $d > 1$ if and only if τ_* is a *multiple* root of $H_m(z, w)$. In this case we say τ_* is a *degenerate direction*.

Take a degenerate direction τ_* . Let us permute the indices of $\{\gamma_j\}$, if necessary, so that $\{\mathcal{G}_{1*}, \dots, \mathcal{G}_{p*}\}$ is the set of all gradient canyons tangent to τ_* , $\mathcal{G}_{k*} := \mathcal{G}_*(\gamma_{k*})$.

Note that these canyons are minimal (Addendum 2.6), hence mutually disjoint.

Theorem D. (See §8 (III).) *Let $\epsilon > 0$ be sufficiently small, $r > 0$, τ_* as above. Then*

$$\int_{I(\gamma_{j*}, d_j - \epsilon, r)} K dS = \int_{\mathcal{G}_{j*}} K dS, \quad d_j := d_{gr}(\gamma_j), \quad 1 \leq j \leq p. \quad (2.17)$$

In a sector around τ_ ,*

$$\lim_{a \rightarrow 0} \int_{I(\tau_*, 1, a)} K dS = \int_{I(\tau_*, 1 + \epsilon, r)} K dS = \sum_{j=1}^p \int_{\mathcal{G}_{j*}} K dS. \quad (2.18)$$

Thus the integral behaves like that of a Dirac function: as $c \rightarrow 0$, the value accumulates over the gradient canyons. We shall refer to this as the *Dirac phenomenon* (§8 (III)).

If $H_m(z, w)$ is *degenerate*, the Dirac phenomenon appears in every degenerate direction. If $H_m(z, w)$ is *non-degenerate*, there is no Dirac phenomenon.

Langevin's Theorem. ([10]) *In an entire neighbourhood of 0,*

$$\lim_{\eta \rightarrow 0} \{ \lim_{c \rightarrow 0} \int_{\mathcal{S}_c \cap D_\eta} K dS \} = 2\pi[\mu_f + (m-1)], \quad \mu_f \text{ the Milnor number of } f.$$

Addendum 2.8. *More specifically, let r be as in (2.5), and let $\{\mathcal{G}_{1*}, \dots, \mathcal{G}_{s*}\}$ be the set of all gradient canyons with gradient degree $d > 1$. Then*

$$\lim_{\eta \rightarrow 0} \{ \lim_{c \rightarrow 0} \int_{\mathcal{S}_c \cap D_\eta} K dS \} = 2\pi m(r-1) + \sum_{i=1}^s \int_{\mathcal{G}_{i*}} K dS.$$

Of course, if H_m is non-degenerate, the right-hand side reduces to $2\pi m(m-1)$.

We now state Theorem E. Let us decompose f_z in $\mathbb{C}\{z, w\}$:

$$f_z(z, w) = \text{unit} \cdot p_1(z, w)^{e_1} \cdots p_s(z, w)^{e_s}, \quad e_k \geq 1,$$

where $p_k(z, w) \in \mathbb{C}\{z, w\}$ are the (distinct) irreducible factors, mini-regular in z .

For convenience, let us permute the indices of $\{\gamma_j\}$, if necessary, so that

$$p_k(\gamma_k(w), w) = 0, \quad 1 \leq k \leq s.$$

That is to say, γ_k , together with the conjugates, are the Newton-Puiseux roots of $p_k(z, w)$.

Take a set of (mutually distinct) *generic* numbers $\epsilon_{k,i}$:

$$\epsilon := \{ \epsilon_{k,i} \mid 1 \leq i \leq e_k, 1 \leq k \leq s, |\epsilon_{k,i}| \text{ sufficiently small} \}.$$

Take a fixed k . Let $\hat{\gamma}_k(y)$ denote $\gamma_k(y)$ with all terms y^e , $e > d_{gr}(\gamma_k)$, deleted. We then perturb $\hat{\gamma}_k(y)$ to

$$\gamma_{k,i}^\epsilon(y) := \hat{\gamma}_k(y) + \epsilon_{k,i} y^{d_k} \in \mathcal{G}(\gamma_k), \quad d_k := d_{gr}(\gamma_k), \quad 1 \leq i \leq e_k.$$

For each i , $\gamma_{k,i}^\epsilon$ generates an *irreducible* function germ

$$p_{k,i}^\epsilon(z, w) \in \mathbb{C}\{z, w\}, \quad p_{k,i}^\epsilon(\gamma_{k,i}^\epsilon(y), y) = 0.$$

We then define

$$P_k^\epsilon(z, w) := \prod_{i=1}^{e_k} p_{k,i}^\epsilon(z, w), \quad f_z^\epsilon(z, w) := \prod_{k=1}^s P_k^\epsilon(z, w).$$

The Newton-Puiseux roots of $f_z^\epsilon(z, w)$ are mutually distinct.

Now let $S^3 \subset \mathbb{C}^2$ denote a 3-sphere centred at 0 with sufficiently small radius,

$$V_{k,i}^\epsilon := \{(z, w) \mid p_{k,i}^\epsilon(z, w) = 0\}, \quad K_{k,i}^\epsilon := V_{k,i}^\epsilon \cap S^3;$$

each $K_{k,i}^\epsilon$ is an iterated torus knot ([3], [11], [14]). We call

$$\mathcal{C}_k^\epsilon := \{K_{k,i}^\epsilon \mid 1 \leq i \leq e_k\}, \quad \mathcal{N}^\epsilon := \{\mathcal{C}_k^\epsilon \mid 1 \leq k \leq s\},$$

a *cable* of knots, and a *network* of cables, respectively. Each \mathcal{C}_k^ϵ lies in a gradient canyon.

Next, take *another* set of generic numbers $\delta := \{\delta_{k,i}\}$. Define $p_{k,i}^\delta, f_z^\delta, K_{k,i}^\delta, \mathcal{C}_k^\delta, \mathcal{N}^\delta$.

Let $\mathcal{L}(K_{k,i}^\epsilon, K_{l,j}^\delta)$ denote the linking number of knots in the usual sense, i.e., the number of crossings needed to separate them. Define the *linking number* of cables by

$$\mathcal{L}(\mathcal{C}_k^\epsilon, \mathcal{C}_l^\delta) := \sum_{i=1}^{e_k} \sum_{j=1}^{e_l} \mathcal{L}(K_{k,i}^\epsilon, K_{l,j}^\delta).$$

Theorem E. Suppose $f(z, w)$ has no multiple root, i.e., $\zeta_i \neq \zeta_j$ if $i \neq j$. Then

$$\mu_f = \mathcal{L}(\mathcal{N}^\epsilon, \mathcal{N}^\delta) := \sum_{k=1}^s \sum_{l=1}^s \mathcal{L}(\mathcal{C}_k^\epsilon, \mathcal{C}_l^\delta).$$

We call \mathcal{N}^ϵ and \mathcal{N}^δ *twin networks*, and $\mathcal{L}(\mathcal{N}^\epsilon, \mathcal{N}^\delta)$ their *linking number*.

3. NEWTON POLYGON RELATIVE TO A POLAR

Take a polar γ which is not a multiple root of $f(z, w)$, i.e., $f(\gamma(y), y) \neq 0$.

We can assume $T(\gamma_*) = [0 : 1]$, so that $\gamma \in \mathbb{D}_{1+}$. We then change variables (formally):

$$Z := z - \gamma(w), \quad W := w, \quad F(Z, W) := f(Z + \gamma(W), W). \quad (3.1)$$

Since $\gamma \in \mathbb{D}_{1+}$, it is easy to see that

$$\|Grad_{z,w} f\| \sim \|Grad_{Z,W} F\|, \quad \Delta_f(z, w) = \Delta_F(Z, W) + \gamma''(W)F_Z^3, \quad (3.2)$$

where $A \sim B$ means $A/B \rightarrow 1$ as $(z, w) \rightarrow 0$.

Recall that the Newton polygon $\mathcal{NP}(F)$ is defined ([6],[7]) as follows. Let us write

$$F(Z, W) = \sum c_{iq} Z^i W^q, \quad c_{iq} \neq 0, \quad i \in \mathbb{Z}, \quad q \in \mathbb{Q},$$

then represent each non-zero monomial term $c_{iq} Z^i W^q$ by a “*Newton dot*” at (i, q) . We shall simply call it a *dot*. The *boundary* of the convex hull generated by these dots in the usual way is the Newton polygon $\mathcal{NP}(F)$, having edges E_i and angles θ_i , as shown in Fig.2.

The *co-slope* of a line passing through $(x, 0)$ and $(0, y)$ is, *by definition*, y/x . Thus

$$\text{co-slope of } E_s = \tan \theta_s.$$

An important step is to study the relationship between the Newton polygons $\mathcal{NP}(F)$ and $\mathcal{NP}(F_Z)$. This is illustrated in Fig.3, which is deliberately drawn off scale for clarity.

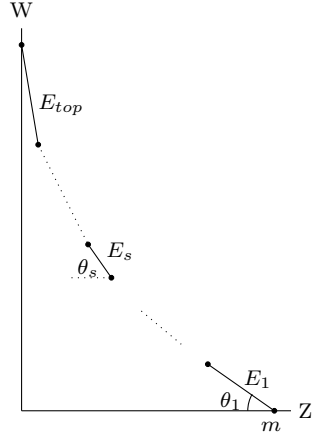
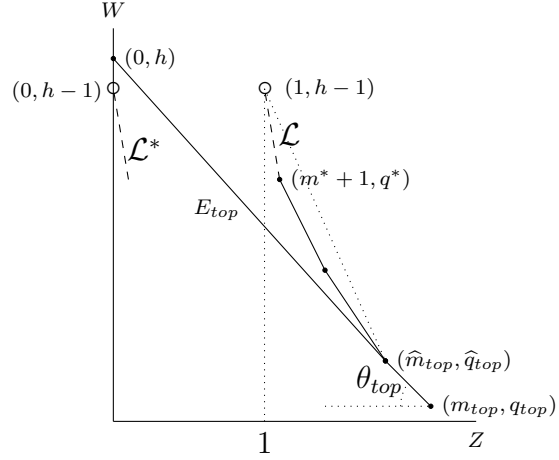
If $i \geq 1$, then (i, q) is a dot of F if and only if $(i-1, q)$ is one of F_Z .

Since γ is a polar, F_Z has no dot of the form $(0, q)$; F has no dot of the form $(1, q)$.

As $f(\gamma(w), w) \neq 0$, we know $F(0, W) \neq 0$. Hence we can write

$$F(0, W) = aW^h + \dots, \quad a \neq 0, \quad h := O_W(F(0, W)) \in \mathbb{Q}. \quad (3.3)$$

Thus $(0, h)$ is a vertex of $\mathcal{NP}(F)$, $(0, h - 1)$ is one of $\mathcal{NP}(F_W)$.

FIGURE 2. $\mathcal{NP}(F)$ FIGURE 3. $\mathcal{NP}(F)$ vs $\mathcal{NP}(F_Z)$.

Let E_{top} denote the top edge of $\mathcal{NP}(F)$, i.e., the edge with left vertex $(0, h)$. Let (m_{top}, q_{top}) denote the right vertex of E_{top} , and θ_{top} the angle of E_{top} , as shown in Fig.3.

Let $(\hat{m}_{top}, \hat{q}_{top}) \neq (0, h)$ be the dot of F on E_{top} which is closest to $(0, h)$. Then, clearly,

$$2 \leq \hat{m}_{top} \leq m_{top}, \quad \frac{h - \hat{q}_{top}}{\hat{m}_{top}} = \frac{h - q_{top}}{m_{top}} = \tan \theta_{top}.$$

Now let us draw a line \mathcal{L} through $(1, h - 1)$ with the following two properties:

- If (m^*, q^*) is a dot of F_Z , then $(m^* + 1, q^*)$ lies on or above \mathcal{L} ;
- There exists a dot (m^*, q^*) of F_Z such that $(m^* + 1, q^*) \in \mathcal{L}$.

Let σ^* denote the co-slope of \mathcal{L} . Let \mathcal{L}^* be the line through $(0, h - 1)$ parallel to \mathcal{L} .

Lemma 3.1. *The number $d_{gr}(\gamma)$ can be computed using the Newton polygons:*

$$d_{gr}(\gamma) = \sigma^* \geq \tan \theta_{top}; \quad \sigma^* = \tan \theta_{top} \iff \tan \theta_{top} = 1. \quad (3.4)$$

No dot of F_W lies below \mathcal{L}^* ; if $\sigma^* > 1$, $(0, h - 1)$ is the only dot of F_W on \mathcal{L}^* .

In Example (2.4), \mathcal{L} is the line joining $(1, n - 1)$ and $(m, 0)$, $d = \frac{n-1}{m-1}$.

Example. For $F(Z, W) = Z^4 + Z^3W^{27} + Z^2W^{63} - W^{100}$ and $\gamma = 0$, $\mathcal{NP}(F)$ has only two vertices $(4, 0)$, $(0, 100)$, while $\mathcal{NP}(F_Z)$ has three: $(3, 0)$, $(2, 27)$, $(1, 63)$. The latter two and $(0, 99)$ are collinear, spanning \mathcal{L}^* ; $h = 100$, $\sigma^* = (99 - 27)/2 = 36$.

Notations. Take $e \geq 1$. Let $\omega(e)$ denote the weight system: $\omega(Z) = e$, $\omega(W) = 1$.

The weighted initial form of $G(Z, W)$ (in the weighted Taylor expansion) is denoted by $\mathcal{I}_{\omega(e)}(G)(Z, W)$, or simply $\mathcal{I}_{\omega}(G)$ when there is no confusion.

Let $\mathcal{I}_{\omega}(G) := \sum a_{ij} Z^i W^{j/N}$. The weighted order of G is $O_{\omega}(G) := ie + j/N$.

Proof. It is clear that $\sigma^* \geq \tan \theta_{top}$, since $(m_{top} - 1, q_{top})$ is a dot of F_Z and $(1, h - 1)$ lies on or above E_{top} .

Moreover, $\sigma^* = \tan \theta_{top}$ if and only if $\tan \theta_{top} = 1$; in this case $d_{gr}(\gamma) = 1$.

Of course, F_W has no dot of the form $(0, q)$, $q < h - 1$.

If (i, q) is a dot of F_W , $i \geq 1$, then $(i, q + 1)$ is one of F , and $(i - 1, q + 1)$ is one of F_Z . Hence all dots of F_W lie on or above \mathcal{L}^* .

Suppose $\sigma^* > 1$. Then $\sigma^* > \tan \theta_{top}$, and hence $(0, h - 1)$ is the only dot of F_W on \mathcal{L}^* .

It remains to show $d_{gr}(\gamma) = \sigma^*$. Note first that since γ is a polar,

$$O_y(\|Grad f(\gamma(y), y)\|) = O_W(F_W(0, W)) = h - 1.$$

Let us first take weight $\omega := \omega(e)$ with $e \geq \sigma^* > 1$. In this case,

$$\mathcal{I}_\omega(F_W)(Z, W) = ahW^{h-1}, \quad O_\omega(F_Z) \geq h - 1;$$

where a, h are as in (3.3), $ah \neq 0$. Hence for all $u \in \mathbb{C}$,

$$O_W(F_W(uW^e, W)) = h - 1, \quad O_W(F_Z(uW^e, W)) \geq h - 1.$$

It follows that $\sigma^* \geq d_{gr}(\gamma)$. Next we show that $\sigma^* > d_{gr}(\gamma)$ is impossible.

Take $\omega(e)$ with $e < \sigma^*$. Since F_Z has a dot on \mathcal{L}^* which is of course not $(0, h - 1)$,

$$O_W(F_Z(uW^e, W)) < h - 1, \quad O_W(\|Grad F(uW^e, W)\|) < h - 1,$$

for generic u . Hence (2.7) is not satisfied. This completes the proof. \square

The following (3.5) is pivotal for proving Theorem A. Note that in (3.2),

$$\Delta_F = -F_{ZZ}F_W^2 - F_Z^2F_{WW} + 2F_ZF_WF_{ZW}.$$

Lemma 3.2. *Take a polar γ , and a weight system $\omega := \omega(e)$.*

If $e \geq d_{gr}(\gamma) > 1$, then the following is true:

$$O_\omega(F_{ZZ}F_W^2) < \min\{O_\omega(F_Z^2F_{WW}), O_\omega(F_ZF_WF_{ZW}), O_\omega(\gamma''F_Z^3)\}. \quad (3.5)$$

It follows that

$$\mathcal{I}_\omega(F_{ZZ}F_W^2) = \mathcal{I}_\omega(\Delta_F) = \mathcal{I}_\omega(\Delta_F + \gamma''F_Z^3). \quad (3.6)$$

If we merely assume $e \geq \tan \theta_1$, then a weaker statement is true:

$$O_\omega(F_{ZZ}F_W^2) = O_\omega(\Delta_F), \quad O_\omega(\Delta_F + \gamma''F_Z^3) = \min\{O_\omega(\Delta_F), O_\omega(\gamma''F_Z^3)\}. \quad (3.7)$$

Proof. First suppose $e \geq \tan \theta_1$. Let us write

$$\mathcal{I}_\omega(F_Z)(Z, W) := a_k Z^k W^q + \cdots + a_M Z^M W^{q-(M-k)e}, \quad a_k a_M \neq 0, \quad (3.8)$$

where $1 \leq k \leq M$. Then, clearly,

$$O_\omega(F_Z) = q + ke, \quad O_\omega(F_{ZZ}) = q + (k - 1)e. \quad (3.9)$$

Now suppose $e \geq \tan \theta_{top}$. Let h, a be as in (3.3). In this case,

$$O_\omega(F) = h, \quad O_\omega(F_W) = h - 1, \quad q + (k + 1)e \geq h. \quad (3.10)$$

The last inequality holds since $(k+1, q)$ lies on or above E_{top} . Thus, by (3.9), (3.10),

$$\begin{aligned} O_\omega(F_{ZZ}F_W^2) &= q + (k-1)e + 2(h-1), \quad O_\omega(\gamma''F_Z^3) = [O(\gamma) - 2] + 3(q+ke), \\ O_\omega(F_{ZZ}F_W^2) &\leq O_\omega(F_Z^2F_{WW}) = O_\omega(F_ZF_WF_{ZW}) = 2(q+ke) + (h-2). \end{aligned} \quad (3.11)$$

Now we can prove (3.5), where $e \geq d_{gr}(\gamma) > 1$. By Lemma 3.1,

$$q + ke \geq h - 1. \quad (3.12)$$

An easy calculation shows that “ \leq ” can be replaced by “ $<$ ” in (3.11), and that

$$O_\omega(F_{ZZ}F_W^2) < O_\omega(\gamma''F_Z^3).$$

This completes the proof of (3.5), and hence also that of (3.6).

We now prove (3.7). First suppose $e \geq \tan \theta_{top}$. In this case we no longer have (3.12). However, the leading term of

$$\mathcal{I}_\omega(F_{ZZ}F_W^2)(Z, W) = ka_k a^2 Z^{k-1} W^{q+2(h-1)} + \dots, \quad (3.13)$$

has Z -order $k-1$. All other terms in $\mathcal{I}_\omega(\Delta_F)$ have Z -order $> k-1$.

Hence no cancellation with the leading term of (3.13) can happen, the first equality in (3.7) follows from (3.11).

The second equality is also clear. The Z -order of $\mathcal{I}_\omega(\gamma''F_Z^3)$ is $3k$, hence

$$O_\omega(\Delta_F) = O_\omega(\gamma''F_Z^3) \implies O_\omega(\Delta_F + \gamma''F_Z^3) = O_\omega(\Delta_F).$$

On the other hand, the following is obviously true:

$$O_\omega(\Delta_F) \neq O_\omega(\gamma''F_Z^3) \implies O_\omega(\Delta_F + \gamma''F_Z^3) = \min\{O_\omega(\Delta_F), O_\omega(\gamma''F_Z^3)\}.$$

Finally, suppose $\tan \theta_1 \leq e < \tan \theta_{top}$. In this case

$$O_\omega(F_W) = q + (k+1)e - 1,$$

where $q \geq 1$ since $(k+1, q)$ can be at worst the left vertex of E_1 . Hence

$$O_\omega(F_{ZZ}F_W^2) = O_\omega(F_ZF_WF_{ZW}) = 3(q+ke) + e - 2 \leq O_\omega(F_Z^2F_{WW}).$$

(The last inequality is an equality if $q > 1$.) The same Z -order argument proves (3.7). \square

Example. It can happen that $\gamma''F_Z^3$ dominates Δ_F . Take

$$f(z, w) := (z - w^2)^m - w^n, \quad \gamma = w^2, \quad e = n/m,$$

where $n > 2m$. We then have

$$O_\omega(F_Z) = \frac{n}{m} \cdot (m-1), \quad O_\omega(F_W) = n-1, \quad O_\omega(\gamma''F_Z^3) < O_\omega(F_{ZZ}F_W^2).$$

4. THE LOJASIEWICZ EXPONENT FUNCTION

Let α be given. Take $e \geq 1$. Take a generic $u \in \mathbb{C}$, or an indeterminate. We write

$$\begin{aligned} |\Delta_f(\alpha(y) + uy^e, y)|^2 &:= N_{(\alpha,e)}(u)y^{2L_\Delta(\alpha,e)} + \dots, \quad N_{(\alpha,e)}(u) \not\equiv 0, \\ \|Grad f(\alpha(y) + uy^e, y)\|^2 &:= D_{(\alpha,e)}(u)y^{2L_{gr}(\alpha,e)} + \dots, \quad D_{(\alpha,e)}(u) \not\equiv 0, \end{aligned} \quad (4.1)$$

where $N_{(\alpha,e)}(u)$, $D_{(\alpha,e)}(u)$ are real-valued, non-negative, polynomials. We also write

$$L_\alpha(e) := L_\Delta(\alpha, e) - 3L_{gr}(\alpha, e), \quad R_{(\alpha,e)}(u) := N_{(\alpha,e)}(u)D_{(\alpha,e)}(u)^{-3}. \quad (4.2)$$

Observe that $L_\Delta(\alpha, e)$, $L_{gr}(\alpha, e)$ are defined even if e is *irrational*. Hence they are *piecewise linear, continuous, increasing* functions of e , defined for $1 \leq e < \infty$.

Convention. We say $\phi(x)$ is *increasing* (resp. *decreasing*, resp. *strictly decreasing*) if

$$x_1 < x_2 \implies \phi(x_1) \leq \phi(x_2) \text{ (resp. } \phi(x_1) \geq \phi(x_2), \text{ resp. } \phi(x_1) > \phi(x_2)).$$

We call $L_\alpha(e)$ the *Lojasiewicz exponent function* along α . It is *piecewise linear, continuous* (but not necessarily increasing),

$$K_*(\eta_*) = 2R_{(\alpha,e)}(u)\delta^{2L_\alpha(e)}, \quad \eta(y) := \alpha(y) + uy^e, \quad u \text{ generic.}$$

Note that $R_\gamma(u)$ and L_γ defined in (2.8) are special cases:

$$R_\gamma(u) = R_{(\gamma,d)}(u), \quad L_\gamma = L_\gamma(d), \quad d := d_{gr}(\gamma).$$

Remark 4.1. As $R_{(\alpha,e)}(u) \neq 0$ for *generic* $u \in \mathbb{C}$, the value $L_*(\eta_*)$ defined in (1.2) is *constant* for generic u . Hence $L_*^{grc}(\mathcal{H}_e)$ in (2.3) is *well-defined*.

Lemma 4.2. *Let γ be a given polar, $1 < d := d_{gr}(\gamma) < \infty$. Then*

- (1) $L_\gamma(e) > -1$ for $1 < e < \tan \theta_1$, θ_1 being the first angle of $\mathcal{NP}(F)$ (Fig.2);
- (2) $L_\gamma(e)$ is *increasing* for $e \in (\tan \theta_1, \tan \theta_{top})$;
- (3) $L_\gamma(e)$ is *decreasing* for $e \in (\tan \theta_{top}, d)$;
- (4) $L_\gamma(e)$ is *strictly decreasing* for $e \in (d - \epsilon, d)$, $\epsilon > 0$ sufficiently small;
- (5) $L_\gamma(d) = -d = L_\gamma$;
- (6) If $O(\alpha - \gamma) \geq d$, then $L_{gr}(\alpha, e) = L_{gr}(\gamma, d)$ and $L_\alpha(e)$ is *increasing* for $e \geq d$.

In particular, since $L_\gamma(e)$ is continuous, the above open intervals can be replaced by closed intervals; $L_\gamma(e) = -d$ is the absolute minimum of $L_\gamma(e)$, $e \in [1, d]$; $L_\gamma(e) \geq -d$ for $e > d$.

Proof. We first prove (6). Let a, h be as in (3.3).

If $O(\alpha - \gamma) \geq d$, then for all $e \geq d$,

$$D_{(\alpha,e)}(u) \geq |ah|^2 > 0 \quad \forall u, \quad L_{gr}(\alpha, e) = L_{gr}(\gamma, d) = h - 1.$$

But $L_\Delta(\alpha, e)$ is increasing, hence so is $L_\alpha(e)$, $e \geq d$.

Let us apply (3.6) with $e = d$. Then (5) follows from:

$$O_{\omega(d)}(F_W) = h - 1, \quad L_\Delta(\gamma, d) = 3(h - 1) - d, \quad L_{gr}(\gamma, d) = h - 1.$$

To prove (4), take weight $\omega(d)$, and write expression (3.8). We then consider $\omega(d - \epsilon)$, where $\epsilon > 0$ is *sufficiently small*. We obviously have

$$\mathcal{I}_{\omega(d-\epsilon)}(F_Z) = a_M Z^M W^{q-(M-k)d}, \quad \mathcal{I}_{\omega(d-\epsilon)}(F_{ZZ}) = a_M M Z^{M-1} W^{q-(M-k)d}.$$

Moreover, (3.6) remains true when $\omega(d)$ is replaced by $\omega(d - \epsilon)$:

$$\mathcal{I}_{\omega(d-\epsilon)}(F_{ZZ} F_W^2) = \mathcal{I}_{\omega(d-\epsilon)}(\Delta_F) = \mathcal{I}_{\omega(d-\epsilon)}(\Delta_F + \gamma'' F_Z^3).$$

It follows that

$$L_\Delta(\gamma, d - \epsilon) = L_\Delta(\gamma, d) - (M - 1)\epsilon, \quad L_{gr}(\gamma, d - \epsilon) = L_{gr}(\gamma, d) - M\epsilon, \quad (4.3)$$

and then

$$L_\gamma(d - \epsilon) - L_\gamma(d) = (2M + 1)\epsilon > 0, \quad L_\gamma(d - \epsilon) > L_\gamma(d).$$

To prove (3), take any e , $\tan \theta_1 < e < d$, $\omega := \omega(e)$. Let us write

$$N := \deg \mathcal{I}_\omega(\Delta_F + \gamma'' F_Z^3)(Z, 1) \quad (\text{degree of a polynomial in } Z). \quad (4.4)$$

Let $\epsilon > 0$ be sufficiently small, then, like (4.3), we have

$$L_\Delta(\gamma, e - \epsilon) = L_\Delta(\gamma, e) - N\epsilon, \quad L_{gr}(\gamma, e - \epsilon) = L_{gr}(\gamma, e) - M\epsilon.$$

Hence it suffices to show that $N \leq 3M$. Let us write $p(Z)$, $p_1(Z)$, $p_2(Z)$ for

$$\mathcal{I}_\omega(F_{ZZ} F_W^2)(Z, 1), \quad \mathcal{I}_\omega(F_Z^2 F_{WW})(Z, 1), \quad \mathcal{I}_\omega(F_Z F_W F_{ZW})(Z, 1)$$

respectively. The first equality in (3.7) implies that

$$\mathcal{I}_\omega(\Delta_F)(Z, 1) = p(Z) + c_1 p_1(Z) + c_2 p_2(Z), \quad c_1, c_2 \in \mathbb{C} \text{ (possibly zero)}.$$

The second equality in (3.7) implies that

$$\mathcal{I}_\omega(\Delta_F + \gamma'' F_Z^3)(Z, 1) = c_3 \mathcal{I}_\omega(\Delta_F)(Z, 1) + c_4 \mathcal{I}_\omega(F_Z^3)(Z, 1), \quad (c_3, c_4) \neq (0, 0).$$

The right hand-side has degree $\leq 3M$. Hence $N \leq 3M$, proving (3).

To prove (2), let us take e , $\tan \theta_1 < e < \tan \theta_{top}$, and write (compare (3.8))

$$\mathcal{I}_\omega(F)(Z, W) = a Z^{M+1} W^l + a' Z^M W^{l+e} + \dots, \quad a \neq 0, \quad \omega = \omega(e).$$

Since $e > \tan \theta_1$, $(M + 1, l)$ cannot be the vertex $(m, 0)$. Hence $l > 0$, and

$$\mathcal{I}_\omega(F_W)(Z, W) = l a Z^{M+1} W^{l-1} + \dots, \quad (4.5)$$

where, since $e < \tan \theta_{top}$, we must have $M + 1 > 0$, and then

$$\mathcal{I}_\omega(F_Z)(Z, W) = a(M + 1) Z^M W^l + \dots. \quad (4.6)$$

Since $e > 1$, an immediate consequence of (4.5) and (4.6) is

$$O_W(F_Z(uW^e, W)) < O_W(F_W(uW^e, W)),$$

and hence

$$O_W\left(\frac{|\gamma'' F_Z^3|^2}{(|F_Z|^2 + |F_W|^2)^3}\right) = O_W(\gamma''), \text{ a constant.}$$

Now, since γ is a polar, we actually have $M + 1 \geq 2$. Hence

$$\mathcal{I}_\omega(F_{ZZ})(Z, W) = aM(M + 1) Z^{M-1} W^l + \dots.$$

Similarly, we can write down the formulas for $\mathcal{I}_\omega(F_{ZW})$ and $\mathcal{I}_\omega(F_{WW})$.

An easy calculation of the determinant Δ_F gives

$$\mathcal{I}_\omega(\Delta_F)(Z, W) = la^3(M+1)(M+l+1)Z^{3M+1}W^{3l-2} + \dots,$$

and then we have

$$\deg \mathcal{I}_\omega(\Delta_F)(Z, 1) = 3M+1 > 3M = \deg \mathcal{I}_\omega(\gamma'' F_Z^3)(Z, 1).$$

It follows from this inequality that

$$\mathcal{I}_\omega(\Delta_F + \gamma'' F_Z^3) = \begin{cases} \mathcal{I}_\omega(\Delta_F) & \text{if } O_\omega(\Delta_F) < O_\omega(\gamma'' F_Z^3), \\ \mathcal{I}_\omega(\gamma'' F_Z^3) & \text{if } O_\omega(\Delta_F) > O_\omega(\gamma'' F_Z^3), \\ \mathcal{I}_\omega(\Delta_F) + \mathcal{I}_\omega(\gamma'' F_Z^3) & \text{if } O_\omega(\Delta_F) = O_\omega(\gamma'' F_Z^3), \end{cases}$$

and hence in every case we have

$$\deg \mathcal{I}_\omega(\Delta_F + \gamma'' F_Z^3)(Z, 1) \geq \deg \mathcal{I}_\omega(F_Z^3)(Z, 1). \quad (4.7)$$

Now, using (4.7) and the same argument as in the proof of (4), we find

$$L_\gamma(e) \geq L_\gamma(e - \epsilon), \quad \epsilon > 0 \text{ sufficiently small.}$$

It remains to prove (1). This case involves only the vertex $(m, 0)$:

$$O_\omega(F) = me, \quad O_\omega(F_Z) = (m-1)e, \quad O_\omega(F_W) > me - 1.$$

It follows that

$$O_\omega(\Delta_F) - 3O_\omega(F_Z) > e - 2 > -1, \quad O_\omega(\gamma'' F_Z^3) - 3O_\omega(F_Z) = O(\gamma'') > -1.$$

This completes the proof of Lemma 4.2. □

Corollary 4.3. *A gradient canyon with $d > 1$ is a curvature tableland.*

5. PROOF OF THEOREMS A AND B

Let us first prove Lemma 2.5. We already know $L_\gamma = -d$ in Lemma 4.2.

First suppose $d := d_{gr}(\gamma) > 1$. In this case $R_\gamma(u)$ is clearly defined for all u since

$$D_{(\gamma, d)}(u) \geq |ah|^2 > 0, \quad a, h \text{ as in (3.3).}$$

Let N, M be as in (3.8) and (4.4), where $e = d$. Then, by (3.5),

$$N = \deg \mathcal{I}_{\omega(d)}(F_{ZZ}F_W^2)(u, 1) = M - 1 < 3M = 3 \deg \mathcal{I}_{\omega(d)}(F_Z)(u, 1).$$

Hence, obviously,

$$\lim_{u \rightarrow \infty} R_\gamma(u) = 0. \quad (5.1)$$

Now suppose $d = 1$. Let us write $H := H_m(z, w)$. By Euler's Theorem,

$$\Delta_H(z, w) = -\frac{m}{m-1} \cdot H \cdot \text{Hess}(H), \quad \text{Hess}(H) := \begin{vmatrix} H_{zz} & H_{zw} \\ H_{wz} & H_{ww} \end{vmatrix}. \quad (5.2)$$

Note that $H_m(z, w)$ has at least two different factors, for otherwise we would have

$$H_m(z, w) = c(z - u_0 w)^m, \quad d_{gr}(\gamma) > 1,$$

a contradiction. It is easy to see that, as a consequence,

$$\text{Hess}(H) \neq 0, \quad \Delta_H \neq 0.$$

Now take any u_0 . If $H(u_0, 1) \neq 0$, then $\text{Grad } H(u_0, 1) \neq 0$, $R_\gamma(u_0)$ is clearly defined.

If $z - u_0 w$ is a factor of H of order k , $k \geq 1$, then it is one of Δ_H of order $3k - 2$,

$$\Delta_H(u, 1) = (u - u_0)^{3k-2} Q_1(u), \quad |\text{Grad } H(u, 1)| = |u - u_0|^{k-1} Q_2(u),$$

where $Q_1(u_0)Q_2(u_0) \neq 0$, u near u_0 . Hence $|u - u_0|^2$ divides $R_\gamma(u)$, $R_\gamma(u_0) = 0$.

Having proved $R_\gamma(u)$ is defined for all u , it remains to show $L_1 = -1$ and (5.1).

We can assume $\gamma \in \mathbb{D}_{1+}$. In terms of (Z, W) we have

$$O_W(\Delta_H) = 3m - 4 < O_W(\gamma'' F_Z^3) = 3(m - 1) + O(\gamma) - 2.$$

Hence

$$O_W(\Delta_H + \gamma'' F_Z^3) = 3m - 4, \quad L_1 = (3m - 4) - 3(m - 1) = -1.$$

We also have (5.1), since

$$\deg \Delta_H(u, 1) \leq 3M - 4 < 3M = 3 \deg H_Z(u, 1).$$

Next we prove Addendum 2.6.

Let γ be a given polar. Consider F , $\mathcal{NP}(F)$, etc. We write $\theta_{\text{top}}(\gamma) := \theta_{\text{top}}$ for clarity.

Suppose $d_{gr}(\gamma) > 1$. If $\mathcal{G}(\gamma_j) \subseteq \mathcal{G}(\gamma)$, then, by Lemma 4.2 (6), $\mathcal{G}(\gamma_j) = \mathcal{G}(\gamma)$. Hence $\mathcal{G}(\gamma)$ is minimal; (2.11) also follows.

By the so-called Kuo-Lu Theorem ([6], Lemma 3.3; [7], Theorem 2.1),

$$\tan \theta_{\text{top}}(\gamma) = \max\{O(\gamma - \zeta_i) \mid 1 \leq i \leq m\}, \quad \zeta_i \text{ as in (2.6)}.$$

Now let r be as in (2.5). By the same theorem, there are exactly $r - 1$ polars γ (counting multiplicities) such that

$$\tan \theta_{\text{top}}(\gamma) = 1, \quad d_{gr}(\gamma) = 1, \tag{5.3}$$

where the second identity follows from the first, by Lemma 4.2.

It is easy to see that (2.12) is true.

Suppose H_m is *degenerate*. We show no minimal canyon with $d = 1$ can exist. By (2.12), it suffices to show that \mathbb{D}_1 is not a minimal canyon.

Take a degenerate direction $\tau_* \in \mathbb{CP}^1$, i.e., a multiple root of H_m . Again, by the same theorem, there is a polar γ_* tangent to τ_* such that

$$\tan \theta_{\text{top}}(\gamma) > 1, \quad d_{gr}(\gamma) > 1.$$

Hence $\mathcal{G}(\gamma)$ is *properly* contained in \mathbb{D}_1 ; \mathbb{D}_1 cannot be a minimal canyon.

On the other hand, if H_m is non-degenerate, every polar γ has $d = 1$, $\mathcal{G}(\gamma) = \mathbb{D}_1$.

Now we prove the first statement of Theorem A.

Let $\mathcal{G}_*(\gamma_*)$, $d := d_{gr}(\gamma) < \infty$, be a minimal gradient canyon. If $d > 1$, it is a curvature tableland by Corollary 4.3. If $d = 1$, then $\mathcal{G}_*(\gamma_*) = \mathbb{CP}_*^1$ by Addendum (2.6).

It remains to show the converse. Let $\mathcal{H}_e(\mu_*)$ be a curvature tableland, μ, e given.

Take a polar γ from the list $\{\gamma_1, \dots, \gamma_{m-1}\}$ in (2.6) such that

$$l := O_y(\mu - \gamma) = \max\{O_y(\mu - \gamma_1), \dots, O_y(\mu - \gamma_{m-1})\}. \quad (5.4)$$

First suppose $e \leq l$. Then, clearly, $\mathcal{H}_e(\mu_*) = \mathcal{H}_e(\gamma_*)$.

If $e > 1$, then by Lemmas 4.2, $\mathcal{H}_e(\gamma_*)$ is a curvature tableland only if $e = d_{gr}(\gamma)$. Hence $\mathcal{H}_e(\mu_*) = \mathcal{G}_*(\gamma_*)$, a minimal canyon.

If $e = 1$, then $\mathcal{H}_1(\mu_*) = \mathbb{C}P_*^1$. This is a curvature tableland only if H_m is non-degenerate. In this case \mathbb{D}_1 is the only gradient canyon, hence minimal.

Now assume $e > l$. We shall show that $\mathcal{H}_e(\mu_*)$ cannot be a curvature tableland.

There are two cases to consider: (i) $l \geq d_{gr}(\gamma)$, (ii) $l < d_{gr}(\gamma)$.

In the case (i), $\mu \in \mathcal{G}(\gamma)$. By Lemma 4.2 (6), L_μ is increasing at every $e > d_{gr}(\gamma)$. Hence condition (2) of Definition 2.1 cannot be satisfied, $\mathcal{H}_e(\mu_*)$ is not a curvature tableland.

In the case (ii), let us write

$$\mu(y) - \gamma(y) := u_0 y^l + \dots, \quad u_0 \neq 0.$$

We can assume $\gamma \in \mathbb{D}_{1+}$. We then consider $F(Z, W)$, $\mathcal{NP}(F)$, $\tan \theta_{top}$, etc., as in (3.1).

Take $\omega := \omega(l)$ and consider $\mathcal{I}_\omega(F_Z)$. An important fact is that we must have

$$\mathcal{I}_\omega(F_Z)(u_0, 1) \neq 0,$$

since otherwise there would be a polar γ_j of the form

$$\gamma_j = \gamma + u_0 y^l + \dots, \quad O_y(\mu - \gamma_j) > l,$$

a contradiction to (5.4). It follows that

$$D_{(\mu, l)}(0) \geq |\mathcal{I}_\omega(F_Z)(u_0, 1)|^2 > 0, \quad L_{gr}(\mu, s) = L_{gr}(\gamma, l) \text{ for all } s \geq l.$$

Now $L_\Delta(\mu, e)$ in (4.1) is an increasing function of e , and $L_{gr}(\mu, s)$ is constant, hence $L_\mu(s)$ is increasing for $s \geq l$. Again condition (2) of Definition 2.1 cannot be satisfied.

The remaining part of Theorem A is readily derived. By (2.8) and (2.9),

$$K_*(\gamma_*^{+c}) = 2R_\gamma(c)\delta^{2L_\gamma}.$$

Hence $R_\gamma(c)$ is a local maximum if and only if (2.4) holds for $\beta_* = \gamma_*^{+c}$.

Corollary 5.1. *Given μ . Choose γ as in (5.4). Then*

$$L_*(\mu_*) \geq -d_{gr}(\gamma),$$

where equality holds only if $\mu_ \in \mathcal{G}_*(\gamma_*)$.*

Next we prove Theorem B.

Let $\mathcal{G}_*(\gamma_*) = \mathcal{H}_d(\gamma_*)$ be a curvature tableland, $R_\gamma(c)$ a local maximum of $R_\gamma(u)$.

Note first that $R_\gamma(c) > 0$ since $R_\gamma(u) \not\equiv 0$. Next, because of (2.10), K_* is not constant on $\mathcal{H}_d(\gamma_*)$. Finally, K_* is constant on $\mathcal{H}_{d+}(\gamma_*^{+c})$ since

$$K_*(\alpha_*) = 2R_\gamma(c)\delta^{2L_\gamma} \quad \text{for all } \alpha_* \in \mathcal{H}_{d+}(\gamma_*^{+c}).$$

Let $\epsilon > 0$ be sufficiently small. Then, clearly,

$$\alpha_* \in \mathcal{H}_d(\gamma_*^{+c}, \epsilon) - \mathcal{H}_{d+}(\gamma_*^{+c}) \implies K_*(\gamma_*^{+c}) > K_*(\alpha_*), \quad L_*(\gamma_*^{+c}) = L_*(\alpha_*). \quad (5.5)$$

Now consider $F(Z, W)$ as in (3.1), and $\mathcal{NP}(F_Z)$, $\mathcal{NP}(F_W)$, $\mathcal{NP}(\Delta_F)$, etc..

Let $\sigma' := \{e_1, \dots, e_s\}$ be the co-slopes of the edges of these Newton polygons. Then,

$$e \notin \sigma' \implies \mathcal{I}_{\omega(e)}(F_Z), \mathcal{I}_{\omega(e)}(\Delta_F), \mathcal{I}_{\omega(e)}(\Delta_F + \gamma'' F_Z^3), \text{ etc., are all monomials.}$$

If $e \notin \sigma'$, then $R_{(\gamma, e)}(u)$ is a *monomial* in u (having $u = 0$ as a pole). Hence

$$e \notin \sigma', u \neq 0 \implies R_{(\gamma, e)}(u) \neq 0.$$

For each $e_i \in \sigma'$, we can find $\epsilon_i > 0$ such that

$$0 < |u| < \epsilon_i \implies R_{(\gamma, e_i)}(u) \neq 0.$$

Now let $\varepsilon := \{\epsilon_1, \dots, \epsilon_s, \epsilon\}$, $\sigma := \sigma' \cup \{d\}$. Then $\mathcal{N}_{\varepsilon, \sigma}^{ptb}(\gamma_*)$ is a neighbourhood with the required properties. (We can discard those $e_i \geq d$.) To see why, let us take $\alpha_* \in \mathcal{N}_{\varepsilon, \sigma}^{ptb}(\gamma_*)$,

$$\alpha(y) - \gamma(y) := ay^e + \dots, \quad a \neq 0.$$

If $e \geq d$, condition (1) follows from (5.5). There is nothing more to prove.

Now, assume $e < d$. To prove condition (2), note that if $e \notin \sigma'$ then

$$R_{(\gamma, e)}(a) \neq 0, \quad K_*(\alpha_*) = 2R_{(\gamma, e)}(a)\delta^{2L_\gamma(e)}.$$

Hence, by Lemma 4.2,

$$L_\gamma(e) > L_*(\mu_*) = L_*(\gamma_*) = -d, \quad K_*(\alpha_*) \ll K_*(\mu_*).$$

If $e = e_i$, the same is true since $0 < |a| < \epsilon_i$. This completes the proof of Theorem B.

6. PROOF OF THEOREMS C AND D

We prove Theorem C. In the first place, along the level curve $F(Z, W) = c$,

$$F_Z dZ + F_W dW = 0, \quad dS = \frac{|F_Z|^2 + |F_W|^2}{|F_W|^2} \frac{dZ \wedge d\bar{Z}}{-2i}. \quad (6.1)$$

Let us take weight $\omega := \omega(d)$, $d := d_{gr}(\gamma)$, and set $Z = uW^d$, then

$$dZ = W^d du + u dW^{d-1} dW.$$

By eliminating dW , we have

$$dZ = \frac{F_W W^d du}{F_W + u dW^{d-1} F_Z}. \quad (6.2)$$

Let h, a be as in (3.3). Then

$$O_W(F_Z(uW^d, W)) = h - 1, \quad F_W(uW^d, W) = haW^{h-1}[1 + o(W)]. \quad (6.3)$$

Convention. As usual, $o(W)$ represents a suitable function $\varphi(u, W)$, $\varphi(u, 0) = 0$.

Since $d - 1 > 0$, (6.2) can be rewritten as

$$dZ = W^d[1 + o(W)]du. \quad (6.4)$$

Let $p(u) := \mathcal{I}_\omega(F_Z)(u, 1)$, $\mathcal{G} := \mathcal{G}(\gamma)$. Then $\deg p(u) = m(\mathcal{G})$, and, by (6.3), (3.6),

$$D_{(\gamma, d)}(u) = |p(u)|^2 + |ha|^2, \quad N_{(\gamma, d)}(u) = |ha|^4 |p'(u)|^2.$$

Now, by (3.6), (6.1) and (6.4), we have

$$\begin{aligned} KdS &= \frac{2|F_{ZZ}F_W^2|^2 + \cdots}{(|F_Z|^2 + |F_W|^2)^2} \cdot \frac{1}{|F_W|^2} \cdot \frac{dZ \wedge d\bar{Z}}{-2i} \\ &= \left\{ \frac{2|p'(u)|^2 |ha|^2}{(|p(u)|^2 + |ha|^2)^2} + o(W) \right\} \frac{du \wedge d\bar{u}}{-2i}. \end{aligned} \quad (6.5)$$

To compute the integral in (2.15) using (6.5), we must know the number of sheets of $\mathcal{S}_c \cap I(\gamma_*, d, R; \eta)$ lying above the u -plane, *i.e.*, the number of values of W when u is given.

For this, let us first consider the special case where all $\gamma_j(y)$ are *integral* power series. In this case there is no conjugation, hence $m(\mathcal{G}) = m(\mathcal{G}_*)$ and

$$h := O_y(f(\gamma(y), y)) = O_W(F(0, W)), \text{ an integer.}$$

When u is given, there are h distinct values of W found by solving the equation

$$c = aW^h + \cdots,$$

where only the higher order terms depend on u . Hence the number of sheets is h .

Thus, when $R > 0$ is given, letting $W \rightarrow 0$, we have

$$\int_{I(\gamma_*, d, R)} KdS = h \cdot \int_{|u| \leq R} \frac{2b^2 |p'(u)|^2}{[b^2 + |p(u)|^2]^2} \frac{du \wedge d\bar{u}}{-2i}, \quad b := |ha|.$$

To compute this integral, let us write

$$p(u) := U(x, y) + iV(x, y), \quad u = x + iy \in \mathbb{C},$$

where U, V satisfy the Cauchy-Riemann equations. Using the latter we have

$$\int \frac{2b^2 |p'(u)|^2}{[b^2 + |p(u)|^2]^2} \frac{du \wedge d\bar{u}}{-2i} = 2b^2 \int \frac{dU \wedge dV}{[b^2 + U^2 + V^2]^2} \quad (\text{indefinite integrals}).$$

Now $u \mapsto p(u)$ is a $m(\mathcal{G})$ -sheet branch covering of \mathbb{C} . Let $R \rightarrow \infty$, we have

$$\int_{\mathbb{C}} \frac{2b^2 |p'(u)|^2}{[b^2 + |p(u)|^2]^2} \frac{du \wedge d\bar{u}}{-2i} = m(\mathcal{G}) \cdot 2b^2 \int_{\mathbb{R}^2} \frac{dU \wedge dV}{[b^2 + U^2 + V^2]^2} = 2\pi m(\mathcal{G}_*),$$

and then (2.15) follows from the identity

$$h m(\mathcal{G}_*) = (h - 1)m(\mathcal{G}_*) + m(\mathcal{G}_*) = \mu_f(\mathcal{G}_*) + m(\mathcal{G}_*).$$

The general case can be derived from the above by the substitution $y \rightarrow y^N$, where N is an integer divisible by all $m_{\text{puis}}(\gamma_j)$ so that all $\gamma_j(y^N)$ are integral power series.

In this way h is magnified to Nh , and \mathcal{S}_c is blown up to N copies of itself. Hence

$$\lim_{R \rightarrow \infty} \int_{I(\gamma_*, d, R)} K dS = \frac{1}{N} \cdot 2\pi(Nh)m(\mathcal{G}_*) = 2\pi[\mu_f(\mathcal{G}_*) + m(\mathcal{G}_*)].$$

To prove (2.16), note that for $I(\gamma_*, d + \epsilon, R; \eta)$ the substitution is $Z = uW^{d+\epsilon}$,

$$dZ = W^{d+\epsilon}[1 + o(W)]du, \quad dZ \wedge d\bar{Z} = |W|^{2d+2\epsilon}[1 + o(W)]du \wedge d\bar{u}.$$

Hence KdS is divisible by $W^{2p\epsilon}$, $p := \deg p(u) \geq 1$. Then (2.16) follows.

We shall point out at the end of this section how to prove (2.16) when $d = 1$.

Convention. We use $C_\phi, C'_\phi, C''_\phi$ to denote *suitable non-zero constants*.

Next we prove Theorem D.

We first prove (2.17); the main idea is exposed here. Write γ_{j*} as γ_* , $d := d_{gr}(\gamma_*)$.

Take weight $\omega := \omega(d - \epsilon)$. Set $Z = uW^{d-\epsilon}$. Notice that

$$(Z, W) \in I(\gamma_*, d - \epsilon, r; \eta) - I(\gamma_*, d, R; \eta) \iff R|W|^\epsilon < |u| \leq r.$$

Since $\epsilon > 0$ is *sufficiently small*, the weighted initial form of F_Z is a monomial:

$$\mathcal{I}_\omega(F_Z)(Z, W) = C_\phi Z^k W^q,$$

where $k \geq 1$ since $F_Z(0, W) = 0$. Hence

$$p(u) := \mathcal{I}_\omega(F_Z)(u, 1) = C_\phi u^k.$$

We also have

$$O_\omega(F_W) = h - 1, \quad O_\omega(F_Z) = (h - 1) - k\epsilon.$$

Now $d > 1$ and ϵ is sufficiently small, hence

$$O_\omega(F_W) < O_\omega(W^{d-\epsilon-1}F_Z), \quad O_W(F_{ZZ}) = O_\omega(F_Z) - (d - \epsilon).$$

Instead of (6.2), we now have

$$dZ = \frac{F_W W^{d-\epsilon} du}{F_W + (d - \epsilon)uW^{d-\epsilon-1}F_Z} = W^{d-\epsilon}[1 + o(W)]du.$$

Instead of (6.5) we have

$$KdS \leq C_\phi |W|^{2k\epsilon} \frac{|u|^{2(k-1)} du \wedge d\bar{u}}{|u|^{4k} - 2i}.$$

The integration is from $|u| = R|W|^\epsilon$ to $|u| = r$. Thus,

$$C_\phi |W|^{2k\epsilon} \cdot 2\pi \int_{R|W|^\epsilon}^r \frac{\rho d\rho}{\rho^{2(k+1)}} = C'_\phi \left[\frac{1}{R^{2k}} - \frac{|W|^{2k\epsilon}}{r^{2k}} \right].$$

As $R \rightarrow \infty$ and $W \rightarrow 0$, the limit is 0. This completes the proof of (2.17).

We now prove (2.18). Let γ_* be a given polar, $d := d_{gr}(\gamma) > 1$, with tangent τ_* .

Let E'_i, θ'_i denote the Newton edges and angles of $\mathcal{NP}(F_Z)$. Let us list the numbers

$$\{\tan \theta'_i \mid \tan \theta'_i \leq d - \epsilon\} \cup \{1, \tan \theta_{top}, d - \epsilon\}$$

(where θ_{top} is the top angle of $\mathcal{NP}(F)$) in ascending order as

$$e_0 := 1 < e_1 < \cdots < e_N := d - \epsilon. \quad (6.6)$$

Lemma 6.1. *Let $\delta > 0$ be sufficiently small, R sufficiently large. Then*

$$\int_{I(\gamma_*, e_i, \delta)} K dS = \int_{I(\gamma_*, e_{i+1}, R)} K dS, \quad 1 \leq i \leq N-1. \quad (6.7)$$

Let $\{c_1, \dots, c_s\}$ be the set of (distinct) roots of $\mathcal{I}_{\omega(e_i)}(F_Z)(z, 1) = 0$, $i \geq 1$. Then

$$\int_{I(\gamma_*, e_i, R)} K dS = \sum_{j=1}^s \int_{I(\gamma_{j*}, e_i, \delta)} K dS, \quad \gamma_j(y) := \gamma(y) + c_j y^{e_i} + \cdots. \quad (6.8)$$

Proof. Take $i \geq 1$. For simplicity, write $e = e_i$, $\hat{e} := e_{i+1}$. Set $Z = uW^e$. As in (6.2),

$$\frac{dZ \wedge d\bar{Z}}{-2i} = \frac{|W|^{2e} |F_W|^2}{|F_W + euW^{e-1}F_Z|^2} \frac{du \wedge d\bar{u}}{-2i}.$$

We then compute the integral of K over $\mathcal{S}_c \cap I(e, \hat{e})$ where

$$I(e, \hat{e}) := \{(Z, W) \in D_\eta \mid R|W|^{\hat{e}} \leq |Z| \leq \delta|W|^e\}.$$

Let us write, as abbreviations,

$$K(A) := \frac{|A|^2}{(|F_Z|^2 + |F_W|^2)^2} \cdot \frac{|W|^{2e}}{|F_W + euW^{e-1}F_Z|^2}, \quad A = \Delta_F \text{ or } \gamma''F_Z^3. \quad (6.9)$$

To show (6.7), it suffices to show that

$$(i) \lim_{W \rightarrow 0} \int_{I(e, \hat{e})} K(\Delta_F) \frac{du \wedge d\bar{u}}{-2i} = 0, \quad (ii) \lim_{W \rightarrow 0} \int_{I(e, \hat{e})} K(\gamma''F_Z^3) \frac{du \wedge d\bar{u}}{-2i} = 0. \quad (6.10)$$

Now write

$$\begin{aligned} \mathcal{I}_{\omega(e)}(F_Z)(Z, W) &= a_k Z^k W^q + a_{k+1} Z^{k+1} W^{q-e} + \cdots, \\ \mathcal{I}_{\omega(\hat{e})}(F_Z)(Z, W) &= \cdots + a_{k-1} Z^{k-1} W^{q+\hat{e}} + a_k Z^k W^q, \end{aligned} \quad (6.11)$$

where $a_k \neq 0$, $k \geq 1$, $k + q\hat{e} < h - 1$ (since $\hat{e} < d_{gr}(\gamma)$).

The following lemma can be proved using the Curve Selection Lemma, or otherwise.

The Vertex Lemma. *Let (s, t) be a vertex of $\mathcal{NP}(G)$ (e.g., $G = F, F_Z$, etc.). Let E_i, E_{i+1} be the adjacent edges with common vertex (s, t) , $e := \tan \theta_i < e' := \tan \theta_{i+1}$. Then*

$$R|W|^{e'} \leq |Z| \leq \delta|W|^e \implies |G(Z, W)| \approx |Z|^s |W|^t.$$

Notation. *We write $G \approx H$ if $C_\phi H \leq G \leq C'_\phi H$.*

We now divide the rest of the proof of (6.7) into two cases.

Case 1: $\tan \theta_{top} \leq e$. (If $\tan \theta_{top} = e_p$ in the list (6.6), this means $p \leq i$.)

In this case $(k+1, q)$ lies on or above (the line extending) E_{top} of $\mathcal{NP}(F)$. Hence

$$e \leq e' \leq \hat{e} \implies q + (k+1)e' \geq h, \quad q + ke' \leq q + k\hat{e} < h - 1. \quad (6.12)$$

Moreover, if $0 \leq |u| \leq \delta$, then

$$|F_W(uW^e, W)| \approx |W|^{h-1}, \quad |F_W + euW^{e-1}F_Z| \approx |F_W|. \quad (6.13)$$

Let us apply the Vertex Lemma to $\mathcal{NP}(F_Z)$ at (k, q) . The following is true:

$$R|W|^{\hat{e}-e} \leq |u| \leq \delta \implies F_Z(uW^e, W) \approx |u|^k |W|^{q+ke}.$$

By (6.12) we also have

$$|F_Z| \geq C_\phi R^k |W|^{q+k\hat{e}} \gg |F_W|, \quad |F_Z|^2 + |F_W|^2 \geq C'_\phi |u|^{2k} |W|^{2(q+ke)}.$$

Similarly, we can show that in $I(e, \hat{e})$,

$$|F_{ZW}| \leq C_\phi |u|^k |W|^{q+ke-1}, \quad |F_{ZZ}| \leq C'_\phi |u|^{k-1} |W|^{q+(k-1)e}.$$

It follows from (3.7) and the above calculations that in $I(e, \hat{e})$,

$$K(\Delta_F) \leq C_\phi |u|^{-2(k+1)} |W|^{2Q(k,e)}, \quad Q(k, e) := (h-1) - (q+ke) > 0.$$

A simple integration gives

$$\int_{I(e, \hat{e})} K(\Delta_F) \frac{du \wedge d\bar{u}}{-2i} \leq 2\pi \{C_\phi W^{2Q(k,e)} - C'_\phi W^{2Q(k, \hat{e})}\}.$$

As $W \rightarrow 0$, the limit is 0, since we also have $Q(k, \hat{e}) > 0$. This proves (6.10)(i).

A similar calculation gives

$$K(\gamma'' F_Z^3) \leq C_\phi |u|^{2k} |W|^{2[O(\gamma)-1]+2[q+(k+1)e-h]},$$

where $O(\gamma) - 1 > 0$ since $\gamma \in \mathbb{D}_{1+}$. Hence (6.10)(ii) follows from (6.12).

Case 2: $e_1 \leq e < \tan \theta_{top}$. In this case (6.13) is no longer true.

We still write (6.11). Then

$$\mathcal{I}_{\omega(e)}(F) = C_\phi Z^{k+1} W^q + \dots, \quad \mathcal{I}_{\omega(e)}(F_W) = q C_\phi Z^{k+1} W^{q-1} + \dots,$$

where $q \geq 1$ since (k, q) can be at worst the left vertex of E_1 . It follows that

$$|F_Z| \approx |u|^k |W|^{q+ke}, \quad |\Delta_F| \leq C_\phi |u|^{3k+1} |W|^{3(q+ke)+e-2}.$$

We also have

$$|F_W + euW^{e-1}F_Z| = C'_\phi |u|^{k+1} |W|^{q+(k+1)e-1} [1 + o(W)].$$

Using (6.9) and the above data, a straightforward computation leads to

$$K(\Delta_F) \leq C_\phi |W|^{2(e-1)}, \quad \lim_{W \rightarrow 0} \int_{I(e, \hat{e})} K(\Delta_F) \frac{du \wedge d\bar{u}}{-2i} = 0. \quad (6.14)$$

Similarly, we have

$$K(\gamma'' F_Z^3) \leq C_\phi |u|^{-2} |W|^{2[O(\gamma)-1]},$$

where $O(\gamma) > 1$. It follows that

$$\lim_{W \rightarrow 0} \int_{I(e, \hat{e})} K(\gamma'' F_Z^3) \frac{du \wedge d\bar{u}}{-2i} = \lim_{W \rightarrow 0} C_\phi |W|^{2[O(\gamma)-1]} \ln |W| = 0. \quad (6.15)$$

It remains to show (6.8). We use the same argument on the compact punched disk

$$D_R^{phd} := \{z \in \mathbb{C} \mid |z| \leq R, |z - c_j| \geq \delta, 1 \leq j \leq s\}.$$

Take $z_0 \in D_R^{phd}$. Let $\epsilon > 0$ be sufficiently small. It suffices to show that

$$\int_{I(\zeta_*, e, \epsilon)} K dS = 0, \quad \zeta(y) := z_0 y^e. \quad (6.16)$$

With the expression (6.11), let us consider the polynomial

$$p(z) := \mathcal{I}_{\omega(e)}(F_Z)(z, 1) = a_k z^k + a_{k+1} z^{k+1} + \dots,$$

and its expansion at z_0 :

$$p(z) = b_0 + b_l(z - z_0)^l + b_{l+1}(z - z_0)^{l+1} + \dots, \quad l \geq 1. \quad (6.17)$$

We must have $b_0 \neq 0$, since otherwise z_0 would be one of the c_j . Hence

$$|F_Z(uW^e, W)| \approx |W|^{q+ke}, \quad |u - z_0| < \epsilon.$$

There are two cases to consider: (a) $e \leq \tan \theta_{top}$, (b) $e > \tan \theta_{top}$.

In (a), we have, by integrating (6.17),

$$\mathcal{I}_{\omega(e)}(F)(z, 1) = b_{-1} + b_0(z - z_0) + \frac{b_l}{l+1}(z - z_0)^{l+1} + \dots. \quad (6.18)$$

Let us first suppose $b_{-1} \neq 0$. In this case

$$|\mathcal{I}_{\omega(e)}(F_W)(uW^e, W)| \approx |W|^{q+(k+1)e-1}, \quad |u - z_0| < \epsilon.$$

We then find, in the same way as before, that

$$K(\Delta_F) \leq C_\phi |W|^{2(e-1)}, \quad K(\gamma'' F_Z^3) \leq C'_\phi |W|^{2[O(\gamma)-1]}.$$

Now $e > 1$ and $O(\gamma) > 1$. A simple integration proves (6.16).

Suppose $b_{-1} = 0$. In this case we have

$$|F_W| \approx |u - z_0| |W|^{q+(k+1)e-1},$$

and then

$$K(\Delta_F) \leq C_\phi |W|^{2(e-1)}, \quad K(\gamma'' F_Z^3) \leq C'_\phi |u - z_0|^{-2} |W|^{2[O(\gamma)-1]},$$

again we have (6.16).

Case (b) is like the above case $b_{-1} \neq 0$. We have

$$K(\Delta_F) \leq C_\phi |W|^{2[(h-1)-(q+ke)]}, \quad K(\gamma'' F_Z^3) \leq C'_\phi |W|^{2[O(\gamma)-1]};$$

again we have (6.16). This completes the proof. \square

Now we can show (2.18). By (2.17) and Lemma 6.1, it suffices to show that

$$\lim_{a \rightarrow 0} \int_{I(0_*, 1, a)} K dS = \int_{I(0_*, e_1, R)} K dS, \quad R \text{ sufficiently large.} \quad (6.19)$$

Since $Z = 0$ is a multiple root of $H_m(Z, W)$, $H_m(Z, W)$ has the form

$$H_m(Z, W) = a_{k+1} Z^{k+1} W^{m-k-1} + \dots + a_0 Z^m, \quad k \geq 1, \quad a_{k+1} \neq 0.$$

In the region $R|W|^{e_1-1} \leq |u| \leq a$, we have

$$|F_Z(uW, W)| \approx |u|^k |W|^{m-1}, \quad |F_W(uW, W)| \leq C_\phi |u|^{k+1} |W|^{m-1}.$$

(The above “ \leq ” can be replaced by “ \approx ” if and only if $m \neq k+1$.) We also have

$$|F_W + uF_Z| \approx |u|^{k+1} |W|^{m-1}, \quad |\Delta_F| \leq C_\phi |u|^{3k+1} |W|^{3m-4}.$$

It follows that in $R|W|^{e_1-1} \leq |u| \leq a$,

$$K(\Delta_F) \leq C_\phi, \quad K(\gamma'' F_Z^3) \leq C'_\phi |u|^{-2} |W|^{2[O(\gamma)-1]},$$

and then we have (6.19).

Finally we prove Addendum 2.8.

The case $r = 1$ is easy: $H_m = C_\phi Z^m$, $O_W(F_W) > m - 1$, and

$$K(\Delta_F) \leq C'_\phi |u|^{-4m} |W|^{2\epsilon}, \quad K(\gamma'' F_Z^3) \leq C''_\phi |u|^{-2} |W|^{2[O(\gamma)-1]},$$

where $\epsilon > 0$. The integrals over $|u| \geq a$, $a > 0$ fixed, have limit 0 as $W \rightarrow 0$.

Now, suppose $r \geq 2$. Using (5.2) and Euler's Theorem, we have

$$\Delta_H(Z, W) = -\frac{mHH_W^2}{W} \frac{\partial}{\partial Z} \left[\frac{H_Z}{H_W} \right], \quad H := H_m(Z, W).$$

On the other hand, with the substitution $Z = uW$, we have, as in (6.2),

$$dZ = \frac{WF_W}{F_W + uF_Z} du = \frac{H_W(u, 1) + \dots}{mH(u, 1) + \dots} du = \left\{ \frac{H_W(u, 1)}{mH(u, 1)} + o(W) \right\} du.$$

Instead of (6.5), we now have

$$KdS = \left\{ \frac{2|Q'(u)|^2}{(1 + |Q(u)|^2)^2} + o(W) \right\} \frac{du \wedge d\bar{u}}{-2i}, \quad Q(u) := \frac{H_Z(u, 1)}{H_W(u, 1)}.$$

If $(Z - cW)^{e+1}$ divides $H(Z, W)$, then $(Z - cW)^e$ divides H_Z and H_W . Hence

$$Q(u) = \frac{p(u)}{q(u)}, \quad \deg p = r - 1 \geq \deg q,$$

where $p(u)$, $q(u)$ are *relatively prime* polynomials. The rational function

$$Q : \mathbb{C} \longrightarrow \mathbb{C}, \quad u \mapsto U + iV := Q(u),$$

is an $(r - 1)$ -fold branch covering, where U, V satisfy the Cauchy-Riemann equations.

Take $\delta > 0$. Consider the punched disk:

$$D_\eta(\delta) := \{(Z, W) \mid |Z - z_j W| \geq \delta |W|, \ 1 \leq j \leq r, \ \sqrt{|Z|^2 + |W|^2} \leq \eta\},$$

where z_j were defined in (2.5). Here, for simplicity, we have assumed $(Z, W) = (z, w)$.

An important observation is that the surface $\mathcal{S}_c \cap D_\eta(\delta)$ consists of m sheets, since for each generic u the surface has m distinct intersecting points with the line $Z = uW$.

Hence, when the integral of K on $\mathcal{S}_c \cap D_\eta(\delta)$ is transformed to one over the complex $(U + iV)$ -plane, the latter ought to be multiplied by a factor of $m(r - 1)$. Thus

$$\lim_{\delta \rightarrow 0} \lim_{W \rightarrow 0} \int_{\mathcal{S}_c \cap D_\eta(\delta)} K dS = m(r - 1) \cdot \lim_{\delta \rightarrow 0} \int_{\mathbb{C}(\delta)} \frac{2dU \wedge dV}{[1 + U^2 + V^2]^2} = 2\pi m(r - 1),$$

where $\mathbb{C}(\delta) := \{z \mid |z - z_j| \geq \delta, 1 \leq j \leq r\}$. This completes the proof of Addendum (2.8).

Langevin's Theorem is a special case. We can prove (2.16) for $d = 1$ in the same way.

7. PROOF OF THEOREM E

It is well-known that one way of computing the Milnor number μ_f is

$$\mu_f = \sum_{j=1}^{m-1} \mu_f(\gamma_j), \quad \mu_f(\gamma_j) := O_w(f_w(\gamma_j, w)).$$

We shall compute each $\mu_f(\gamma_j)$.

For convenience, let us re-name and list the roots of $f_z^\epsilon(z, w)$ and $f_z^\delta(z, w)$ as

$$\{\gamma_1^\epsilon, \dots, \gamma_{m-1}^\epsilon\} \quad \text{and} \quad \{\gamma_1^\delta, \dots, \gamma_{m-1}^\delta\}$$

respectively, in such a way that

$$d_{gr}(\gamma_j) = O(\gamma_j - \gamma_j^\epsilon) = O(\gamma_j - \gamma_j^\delta) = O(\gamma_j^\epsilon - \gamma_j^\delta), \quad 1 \leq j \leq m - 1. \quad (7.1)$$

Lemma 7.1. *For $1 \leq j, k \leq m - 1$,*

$$O(\gamma_j^\epsilon - \gamma_k^\delta) = \begin{cases} d_{gr}(\gamma_j) = d_{gr}(\gamma_k) & \text{if } \mathcal{G}(\gamma_j) = \mathcal{G}(\gamma_k), \\ O(\gamma_j - \gamma_k) & \text{otherwise.} \end{cases} \quad (7.2)$$

Proof. First, suppose

$$\mathcal{G}(\gamma_j) \neq \mathcal{G}(\gamma_k), \quad d_{gr}(\gamma_j) > 1, \quad \mathcal{G}(\gamma_k) > 1.$$

In this case, by Addendum 2.6, $\mathcal{G}(\gamma_j)$, $\mathcal{G}(\gamma_k)$ are minimal, hence disjoint,

$$d_{gr}(\gamma_j) > O(\gamma_j - \gamma_k) < d_{gr}(\gamma_k), \quad O(\gamma_j - \gamma_k) = O(\gamma_j^\epsilon - \gamma_k^\delta).$$

Now suppose

$$d_{gr}(\gamma_j) = 1 < d_{gr}(\gamma_k).$$

In this case,

$$\mathbb{D}_1 = \mathcal{G}(\gamma_j) \supset \mathcal{G}(\gamma_k), \quad O(\gamma_j - \gamma_k) = 1 = O(\gamma_j^\epsilon - \gamma_k^\delta).$$

The rest of the proof is easy. □

Let γ_j be given. To compute $\mu(\gamma_j)$, let us first assume $d := d_{gr}(\gamma_j) > 1$.

We consider $\mathcal{NP}(F)$, $\mathcal{NP}(F_Z)$ for $\gamma := \gamma_j$. As pointed out in §3, if $E_t := E_{top}$ is the top edge of $\mathcal{NP}(F)$, then for each $s \leq t - 1$, the edge E'_s of $\mathcal{NP}(F_Z)$ is obtained by moving E_s to the left by 1. For $s \geq t$, the situation is illustrated in Fig.3.

The line \mathcal{L}^* in Fig. 3, of co-slope d , meets $\mathcal{NP}(F_Z)$ either along an edge or at a vertex. Let E'_r be the edge whose right vertex (m'_r, q'_r) lies on \mathcal{L}^* , where, of course, $r \geq t$.

Let σ_k^* denote the co-slope of E'_k . Then, by the Kuo-Lu theorem,

$$\#\{i \mid O(\gamma_j - \gamma_i) \geq d\} = m'_r, \quad \#\{i \mid O(\gamma_j - \gamma_i) = \sigma_k^*\} = m'_k - m'_{k+1}, \quad (7.3)$$

where $1 \leq k \leq r-1$, $m'_r = m(\mathcal{G}_j)$. Note also that

$$\mu_f(\gamma_j) = O_w(f(\gamma_j, w)) - 1 = h - 1, \quad h \text{ as in (3.3)}.$$

Now, as can be seen from $\mathcal{NP}(F_Z)$,

$$h - 1 = m'_r d + (m'_{r-1} - m'_r) \sigma_{r-1}^* + \cdots + (m'_1 - m'_2) \sigma_1^*, \quad m'_1 = m - 1. \quad (7.4)$$

By Lemma 7.1,

$$m'_r = \sharp\{i \mid O_y(\gamma_j^{(\epsilon)} - \gamma_i^{(\delta)}) = d\}, \quad m'_k - m'_{k+1} = \sharp\{i \mid O(\gamma_j^{(\epsilon)} - \gamma_i^{(\delta)}) = \sigma_k^*\},$$

where $1 \leq k \leq r-1$. It follows from (7.4) that

$$\mu_f(\gamma_j) = \sum_{k=1}^{m-1} O_y(\gamma_j^{(\epsilon)} - \gamma_k^{(\delta)}). \quad (7.5)$$

Now, assume $d_{gr}(\gamma_j) = 1$. In this case,

$$\tan \theta_{top} = 1, \quad \mu_f(\gamma_j) = m - 1, \quad O_y(\gamma_j^{(\epsilon)} - \gamma_k^{(\delta)}) = 1,$$

where $1 \leq k \leq m-1$. Hence (7.5) remains true, and then

$$\mu_f = \sum_{j=1}^{m-1} \mu_f(\gamma_j) = \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} O_y(\gamma_j^{(\epsilon)} - \gamma_k^{(\delta)}) = \mathcal{L}(\mathcal{N}_\varepsilon, \mathcal{N}_\delta).$$

8. NOTES

(I) We give a proof of (1.1). (Compare [12].) First, suppose $f = c$ is a graph:

$$f(z, w) - c = w - g(z), \quad g(z) = u(x, y) + iv(x, y),$$

where $g(z)$ is holomorphic, $z = x + iy$. For the First Fundamental Form, we have

$$E = G = 1 + |g'(z)|^2 = 1 + u_x^2 + v_x^2, \quad F = 0;$$

and the (negative of the usual) Gaussian curvature ([2], p.237) is

$$K = \frac{1}{2E^3} \cdot [E(E_{xx} + E_{yy}) - (E_x^2 + E_y^2)].$$

Using the Cauchy-Riemann equations, we then have

$$K = \frac{2}{E^3} [u_{xx}^2 + v_{xx}^2] = \frac{2}{E^3} |g''(z)|^2 = \frac{2|\Delta_f|^2}{\|Grad f\|^6}.$$

Now the general case. Near a regular point (z_0, w_0) of $f(z, w) = c$, we can write

$$f(z, w) - c = \mu(z, w)[(w - w_0) - g(z - z_0)], \quad \mu(z_0, w_0) \neq 0 \quad (\mu \text{ a unit}).$$

We then *evaluate the derivatives at* (z_0, w_0) : $f_z = -\mu g'$, $f_w = \mu$ and

$$f_{zz} = -2\mu_z g' - \mu g'', \quad f_{ww} = 2\mu_w, \quad f_{zw} = -\mu_w g' + \mu_z,$$

whence $\Delta_f(z_0, w_0) = \mu(z_0, w_0)^3 g''(z_0)$. This completes the proof.

(II) A necessary and sufficient condition for $K_* = \text{const}$ on \mathbb{CP}_*^1 is that

$$f(z, w) = \text{unit} \cdot [z - \zeta(w)]^m, \quad \zeta(w) \text{ an integral power series.}$$

For if not all ζ_i in (2.6) are equal, there would exist a polar γ , $f(\gamma, w) \neq 0$, $d < \infty$.

(III) The intervals $\mathcal{H}_e(\alpha_*, r)$ generate a σ -algebra in \mathbb{C}_* (Measure Theory). We define

$$\mu(\mathcal{H}_e(\alpha_*, r)) := \int_{I(\alpha_*, e, r)} K dS,$$

and then extend μ to a *measure* on the σ -algebra. Thus,

$$\mu(\mathcal{H}_e(\alpha_*)) = \lim_{R \rightarrow \infty} \int_{I(\alpha_*, e, R)} K dS, \quad \mu(\mathcal{G}_*) = \int_{\mathcal{G}_*} K dS \quad (\text{as in (2.13)}).$$

Theorem D (restated). *Let \mathcal{H}_e be a given horn subspace, $e > 1$. Let $\{\mathcal{G}_{1*}, \dots, \mathcal{G}_{s*}\}$ be the set of all gradient canyons contained in \mathcal{H}_e . Then*

$$\mu(\mathcal{H}_e) = \sum_{j=1}^s \mu(\mathcal{G}_{j*}). \quad (\text{The Dirac Phenomenon})$$

If no \mathcal{G}_{j} exists, the right-hand side is of course zero.*

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